A probabilistic numerical method for optimal multiple switching problem and application to investments in electricity generation

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Abstract

In this paper, we present a probabilistic numerical algorithm combining dynamic programming, Monte Carlo simulations and local basis regressions to solve non-stationary optimal multiple switching problems in infinite horizon. We provide a comprehensive rate of convergence of the method in terms of the time step used to discretize the problem, of the size of the local hypercubes involved in the regressions, and of the truncating time horizon. To make the method viable for problems in high dimension and long time horizon, we extend a memory reduction method to the general Euler scheme, so that, when performing the numerical resolution, the storage of the Monte Carlo simulation paths is not needed. Then, we apply this algorithm to a model of optimal investment in power plants. This model takes into account electricity demand, cointegrated fuel prices, carbon price and random outages of power plants. It computes the optimal level of investment in each generation technology, considered as a whole, w.r.t. the electricity spot price. This electricity price is itself constructed according to an extended structural model. In particular, it is a function of the random processes as well as the installed capacities. The evolution of the optimal generation mix is illustrated on a realistic numerical problem in dimension eight, i.e. with two different technologies and six random processes.

1 Introduction

This paper presents a probabilistic numerical method for multiple switching problem with an efficient application to a new stylized long-term investment model for electricity generation. Since electricity cannot be stored and building new plants takes several years, investment in new capacities must be decided a long time in advance if a country wishes to be able to satisfy its demand[1]. Before the worldwide liberalization of the electricity sector, electric utilities were monopolies whose objective was to plan the construction of power plants in order to satisfy demand at the least cost within a given loss of load probability or an accepted level of energy non-served. This investment process was called *generation expansion planning* (GEP). Its output was mostly a given set of power plants to build for the next ten or twenty years (see International Atomic Energy Agency [1984] for a comprehensive description of the GEP methodology and difficulties). In spite of thirty years of liberalization of the electricity sector, of the recognition that GEP methods were inadequate within a market context (Hobbs [1995], Dyner and Larsen [2001]) and of an important set of alternative methods (see Foley et al. [2010] and Connolly et al. [2010] for recent surveys on generation investment models and softwares), power utilities still heavily rely on GEP methods (see Aïd [2010]). Real options methods, which should have been the natural alternative valuation method for firms converted to a

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[1]See for instance the recent impressive Indian electricity system blackout of July 30st, and August 1st, 2012.

value maximizing objective, did not emerge as the method of choice. Despite the important body of literature that followed McDonald and Siegel [1986]'s seminal paper, Dixit and Pindyck [1994]'s monography and implementations for electricity generation investment (see for instance Botterud et al. [2005]), real options still remain a marginal way of assessing investment decisions both in the electric sector and in the industry in general (see for instance, amongst the recurrent surveys on capital budgeting methods, Baker et al. [2012]'s recent survey).

The main reason for this situation lies in the considerable mathematical difficulties involved in the conception of a tractable yet realistic real options model for electricity generation. On the one hand, if one chooses an exogenous electricity spot price model (price-taking firm situation), one gets a low-dimensional model that should be numerically and maybe even analytically tractable (see for instance Botterud et al. [2005]). But, in this case, one would fail to account for the long-term impact of investment decisions on spot prices. In the other hand, if one wishes to capture the long-term effect of investment decisions on spot prices, one would run into the curse of dimensionality in the face of too high-dimensional a problem to be tracted neither analytically nor numerically. Because of the dimensionality problem, the main alternative proposed to take into account uncertainties is stochastic programming methods where random events are represented with a tree (see Ahmed et al. [2003] for a recent typical implementation of this approach). The resulting tractability is however obtained at the expense of a crude simplification of the statistical properties of the stochastic processes of the underlying random events. To the knowledge of the authors, the only reported attempt to apply a full stochastic dynamic approach to investment decision can be found in Mo et al. [1991], where stochastic dynamic programming is used to solve an investment problem in dimension four (demand, oil price, installed capacity and year) with prices modelled as Markov chains.

Our approach in this paper takes advantage of the considerable progress made in the last ten years by numerical methods for high-dimensional American options valuation problems to propose a probabilistic way to look at future electricity generation mixes. For an up-to-date state of the art on this subject, the reader is referred to the recent book Carmona et al. [2012].

In this paper, we first adapt the resolution of American options problems by Monte-Carlo methods, popularized by Longstaff and Schwartz [2001], Tsitsiklis and Van Roy [2001], to the more general class of optimal switching problems. The crucial choice of regression basis is done here in the light of the work of Bouchard and Warin [2011], so as to obtain a stable algorithm suited to high-dimensional problems, aiming at the best possible numerical complexity. The memory complexity, often acknowledged as the major drawback of this Monte Carlo approach (see Carmona and Ludkovski [2008]), is drastically slashed by generalizing the memory reduction method from Chan et al. [2004, 2006], Chan and Wu [2011] to any stochastic differential equation. We provide a rigorous and comprehensive analysis of the rate of convergence of our algorithm, taking advantage of the works of, most notably, Bouchard and Touzi [2004], Tan [2011] and Gassiat et al. [2012]. Note that such unusual features as infinite horizon and non-stationarity are encompassed here. Finally, we build a long-term structural model for the spot price of electricity, extending the work of Aïd et al. [2009] and Aïd et al. [2012] in several directions (cointegrated fuels and CO_2 prices, stochastic availability rate of production capacities, etc.). This model is itself incorporated into an optimal control problem corresponding to the search for the optimal investments in electricity generation. The resolution of this problem using our algorithm is illustrated on a simple numerical example with two different technologies, leading to an eight-dimensional problem (demand, CO_2 price, and, for each technology, fuel price, random outages and the controlled installed capacity). The time evolution of the distribution of power prices and of the generation mix is illustrated on a forty-year time horizon.

To sum up, the contribution of the paper is threefold. Firstly, it provides, for a wisely chosen regression basis, a comprehensive analysis of convergence of a regression-based Monte-Carlo algorithm for a class of infinite horizon optimal multiple switching problems, large enough to handle realistic short term profit functions and investment cost structures with possible seasonality patterns. Secondly, it adapts and generalizes a memory reduction method in order to slash the amount of memory required by the Monte Carlo algorithm. Thirdly, a new stylised investment model for electricity generation is proposed, taking into account electricity demand, cointegrated fuel prices, carbon price and random outages of power plants, used as building blocks of a new structural model for the electricity spot price. A numerical resolution of this investment problem with our algorithm is illustrated on a specific example, providing, among many other outputs, an electricity spot price dynamics consistent with the investment decision process in power generation.

The outline of the paper is the following. Section 2 describes the class of optimal switching problems studied here, including the detailed list of assumptions considered. Section 3 describes the resolution algorithm and analyzes its rate of convergence. Section 4 details the computational complexity of the algorithm, as well as its memory complexity, along with the construction of the memory reduction method. Finally, Section 5 introduces the extended structural model of power spot price, the investment problem, as well as an illustrated numerical resolution. Section 6 concludes the paper.

Notations

Here are a few notations that will be used throughout the paper:

- The notation $\mathbf{1}\{.\}$ stands for the indicator function.
- Throughout the paper, $C > 0$ may denote a generic constant, independent of the surrounding variables, whose value may differ from line to line (usually between two successive inequalities in a given proof).
- For any stochastic process $X = (X_s)_{s\geq 0}$ taking values in a given set $\mathcal{X}$, and any $(t,x) \in \mathbb{R}^+ \times \mathcal{X}$, we denote as $X^{t,x} = (X_s^{t,x})_{s\geq t}$ the stochastic process with the same dynamics as X, but starting from x at time t: $X_t^{t,x} = x$.
- For any $(a,b) \in \mathbb{R}\times\mathbb{R}$, $a \wedge b := \min(a,b)$ and $a \vee b := \max(a,b)$.

2 Optimal switching problem

2.1 Formulation

Fix a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$, where $\mathbb{F}$ satisfies the usual conditions. We consider the following general class of (non-stationary) optimal switching problems:

$$v(t,x,i) = \sup_{\alpha\in\mathcal{A}_{t,i}} \mathbb{E}\left[\int_t^{+\infty} f\left(s, X_s^{t,x}, I_s^\alpha\right) ds - \sum_{\tau_n \geq t} k\left(\tau_n, \zeta_n\right)\right] \tag{2.1}$$

where:

- $X^{t,x} = (X_s^{t,x})_{s\geq t}$ is an $\mathbb{R}^d$-valued, $\mathbb{F}$-adapted markovian diffusion starting from $X_t = x$, with generator $\mathcal{L}$.
- $I^\alpha = (I_s^\alpha)_{s\geq 0}$ is a càd-làg, $\mathbb{R}^{d'}$-valued, $\mathbb{F}$-adapted piecewise constant process. It is controlled by a strategy α, described below. We suppose it can only take values into a fixed finite set $\mathbb{I}_q = \{i_1, i_2, \ldots, i_q\}$, $q \in \mathbb{N}^*$, which means that equation (2.1) corresponds to an optimal switching problem.
- An impulse control strategy α corresponds to a sequence $(\tau_n, \iota_n)_{n\in\mathbb{N}}$ of increasing stopping times $\tau_n \geq 0$, and $\mathcal{F}_{\tau_n}$-measurable random variables ι_n valued in $\mathbb{I}_q$. Using this sequence, $I^\alpha = (I_s^\alpha)_{s\geq 0}$ is defined as follows:

  $$I_s^\alpha = \iota_0 \mathbf{1}\{0 \leq s < \tau_0\} + \sum_{n\in\mathbb{N}} \iota_n \mathbf{1}\{\tau_n \leq s < \tau_{n+1}\} \in \mathbb{I}_q$$

  Alternatively, α can be described by the sequence $(\tau_n, \zeta_n)_{n\in\mathbb{N}}$, where $\zeta_n := \iota_n - \iota_{n-1}$ (and $\zeta_0 := 0$). Using this alternative sequence, I^α can be written as follows:

  $$I_s^\alpha = \iota_0 + \sum_{\tau_n \leq s} \zeta_n \in \mathbb{I}_q$$

- $\mathcal{A}$ is the set of admissible strategies. A strategy α belongs to $\mathcal{A}$ if $\tau_n \to +\infty$ a.s. as $n \to \infty$.
- For any $(t,i) \in \mathbb{R}^+ \times \mathbb{I}_q$, the set $\mathcal{A}_{t,i} \subset \mathcal{A}$ is defined as the subset of admissible strategies α such that I^α starts at time t from the value i.
- f and k are $\mathbb{R}$-valued measurable functions.

2.2 Assumptions

We complete the above formulation with the following relevant assumptions.

Assumption 1. *[Diffusion] The $\mathcal{F}_t$-adapted, $\mathbb{R}^d$-valued uncontrolled process X is a diffusion process, governed by the dynamics*

$$\begin{aligned} dX_s &= b(s, X_s)\, ds + \sigma(s, X_s)\, dW_s \\ X_0 &= x_0 \in \mathbb{R}^d \end{aligned} \tag{2.2}$$

where W is an $\mathcal{F}_t$-adapted standard d-dimensional Brownian motion, and b and σ are respectively $\mathbb{R}^d$-valued and $\mathbb{R}^{d\times d}$-valued function.

Assumption 2. *[Lipschitz] The functions $b : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are Lipschitz-continuous (uniformly in t) with linear growth: $\exists C_b, C_\sigma > 0$ s.t. $\forall t \in \mathbb{R}^+$, $\forall (x,x') \in \left(\mathbb{R}^d\right)^2$:*

$$\begin{aligned} |b(t,x) - b(t,x')| &\leq C_b |x - x'| \\ |b(t,x)| &\leq C_b (1 + |x|) \\ |\sigma(t,x) - \sigma(t,x')| &\leq C_\sigma |x - x'| \\ |\sigma(t,x)| &\leq C_\sigma (1 + |x|) \end{aligned}$$

Remark 2.1. Assumption 2 is sufficient to prove the existence and unicity of a strong solution to the SDE (2.2) (see for instance Theorem 5.2.1 in Øksendal [2007]).

Assumption 3. *[Discrete and bounded controls] The increasing, càd-làg, $\mathcal{F}_t$-adapted, $\mathbb{R}^{d'}$-valued controlled process I^α, $\alpha \in \mathcal{A}$, takes its values into a fixed finite set $\mathbb{I}_q = \{i_1, i_2, \ldots, i_q\} \in \left(\mathbb{R}^{d'}\right)^q$, $q \in \mathbb{N}^*$, with $i_0 = 0$ $\left(\in \mathbb{R}^{d'}\right)$ and such that all the elements in $\mathbb{I}_q$ are different.*

Assumption 4. *[Lipschitz&Discount] The functions f and k decrease exponentially in time: $\exists \rho > 0$ s.t. $\forall (t,x,i,j) \in \mathbb{R}^+ \times \mathbb{R}^d \times (\mathbb{I}_q)^2$:*

$$\begin{aligned} f(t,x,i) &= e^{-\rho t} \tilde{f}(t,x,i) \\ k(t,j-i) &= e^{-\rho t} \tilde{k}(t,j-i) \end{aligned}$$

where the functions $\tilde{f}$ and $\tilde{k}$ are Lipschitz continuous with linear growth:

$\exists C_f, C_k > 0$ s.t. $\forall \{(t,x,i,j),(t',x',i',j')\} \in \left\{\mathbb{R}^+ \times \mathbb{R}^d \times (\mathbb{I}_q)^2\right\}^2$:

$$\begin{aligned} \left|\tilde{f}(t,x,i) - \tilde{f}(t',x',i')\right| &\leq C_f (|t-t'| + |x-x'| + |i-i'|) \\ \left|\tilde{f}(t,x,i)\right| &\leq C_f (1 + |x|) \\ \left|\tilde{k}(t,j-i) - \tilde{k}(t',j'-i')\right| &\leq C_k (|t-t'| + |(j-i) - (j'-i')|) \end{aligned}$$

Moreover, we assume in the following that $\rho > \max\left(\rho_1, C_b + \frac{1}{2}C_\sigma^2\right)$, (see equation (2.3) where ρ_1 is defined, and Remark 3.5).

Assumption 5. *[Fixed costs] The cost function $k : \mathbb{R}^+ \times \mathbb{R}^{d'} \to \mathbb{R}^+$ is such that:*

- $\forall t \in \mathbb{R}^+,\ k(t,0) = 0$
- $\exists \kappa > 0$ s.t. $\forall t \in \mathbb{R}^+,\ \forall (i,j) \in (\mathbb{I}_q)^2,\ \{i \neq j\} \Rightarrow \left\{\tilde{k}(t, j-i) \geq \kappa\right\}$
- *(triangular inequality)* $\forall t \in \mathbb{R}^+,\ \forall (i,j,k) \in (\mathbb{I}_q)^3$ *with* $i \neq j$ *and* $j \neq k$*:*
$$k(t, k-i) < k(t, j-i) + k(t, k-j)$$

Remark 2.2. The economic interpretations of Assumption 5 are the following:

1. There is no cost for not switching, but any switch incurs at least a positive fixed cost.
2. At any given date, it is always cheaper to switch directly from i to k than to switch first from i to j and then from j to k.

Remark 2.3. Assumption 5 is required for an optimal strategy to be unique, and enables the simplification from (2.8) to (2.11).

Remark 2.4. Under those standard assumptions, the value function v is well-defined and finite. Indeed, recall first that under Assumption 2, there exists, for every $p \geq 1$, positive constants C_p and ρ_p such that $\forall s \geq t \geq 0$ and $\forall x \in \mathbb{R}^d$:

$$\mathbb{E}\left[\left|X_s^{t,x}\right|^p\right] \leq C_p\left(1 + |x|^p\right)\exp\left(\rho_p\left(s-t\right)\right) \tag{2.3}$$

(use Burkholder-Davis-Gundy inequality and Gronwall's Lemma, see for instance Kloeden and Platen [1999] Theorem 4.5.4 for the even power case). Hence, $\forall (t_0, t, x, i) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^{d'}$ with $t_0 \leq t$ and $\forall \alpha \in \mathcal{A}_{t_0,i}$:

$$\begin{aligned}
\mathbb{E}\left[\int_t^{+\infty} \left|f\left(s, X_s^{t_0,x}, I_s^\alpha\right)\right| ds\right] &\leq C_f \int_t^{+\infty} e^{-\rho s}\left(1 + \mathbb{E}\left[\left|X_s^{t_0,x}\right|\right]\right) ds \\
&\leq C_f\left(e^{-\rho t} + (1+|x|)\int_t^{+\infty} e^{-\rho s} e^{\rho_1 (s-t_0)} ds\right) \\
&\leq C_f\left(1+|x|\right) e^{-\bar{\rho} t - \rho_1 t_0}
\end{aligned} \tag{2.4}$$

where $\bar{\rho} := \rho - \rho_1 > 0$ (Assumption 4). In particular, the costs being positive (Assumption 5), and recalling (2.1), it holds that:

$$v(t,x,i) \leq C_f\left(1+|x|\right) e^{-\rho t} \tag{2.5}$$

2.3 Outline of the resolution

From a theoretical point of view, the value functions $v_i := v(.,.,i)$, $i \in \mathbb{I}_q$ from equation (2.1) are known to satisfy (under suitable conditions on $f_i(.,.) := f(.,.,i)$ and k, see for instance Seydel [2009] in a much more general setting) the following Hamilton-Jacobi-Bellman Quasi-Variational Inequalities (HJBQVI):

$$\min\left\{-\frac{\partial v_i}{\partial t}(t,x) - \mathcal{L} v_i(t,x) - f_i(t,x)\ ,\ v_i(t,x) - \max_{j \in \mathbb{I}_P,\, j \neq i}\left(v_j(t,x) - k(t, j-i)\right)\right\} = 0\,,\ \forall (t,x,i) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{I}_q \tag{2.6}$$

together with suitable limit condition.

Alternatively, $v(t, X_t, i)$ can be characterized as the solution of a particular Reflected Backward Stochastic Differential Equation (Hamadène et al. [1999], El Asri [2010]).

Moreover, (2.1) satisfies the well-known dynamic programming principle, i.e., for any stopping time $\tau \geq t$:

$$v(t,x,i) = \sup_{\alpha \in \mathcal{A}_{t,i}} \mathbb{E}\left[\int_t^\tau f\left(s, X_s^{t,x,}, I_s^\alpha\right) ds - \sum_{t \leq \tau_n \leq \tau} k\left(\tau_n, \zeta_n\right) + v\left(\tau, X_\tau^{t,x}, I_s^\alpha\right)\right] \tag{2.7}$$

From a practical point of view, apart from a few simple examples in low-dimension, finding directly the solution of the HJBQVI (2.6) is usually infeasible, and the numerical PDE tools become cumbersome and inefficient in the multi-dimensional setting. Instead, probabilistic methods that make use of (2.7), in the spirit of Carmona and Ludkovski [2008], are usually more practical and versatile.

Indeed, as the diffusion X is not controlled, this optimal switching problem can be seen as an extended American option problem. This suggests that, up to some adjustments, the probabilistic numerical tools developed in this context (see Bouchard and Warin [2011] for instance) may be adapted to solve (2.1).

To be more specific, consider a variant $\hat{v}$ of (2.1) such that the switching decisions can only take place on a finite time grid $\Pi = \{t_0 = 0 < t_1 < \ldots < t_m = T\}$ for a fixed $T > 0$. Then $\forall i \in \mathbb{I}_q$, $\forall x \in \mathbb{R}^d$, and $\forall t_k \in \Pi$, the dynamic programming principle (2.7) becomes:

$$\hat{v}_i(t_k, x) = \max\left\{E_i(t_k, x), \max_{j \in \mathbb{I}_q,\, j \neq i} \{\hat{v}_j(t_k, x) - k(t_k, j - i)\}\right\} \tag{2.8}$$

where:

$$E_i(T, x) := \mathbb{E}\left[\int_T^{+\infty} f_i\left(s, X_s^{T,x}\right) dt\right] \tag{2.9}$$

$$E_i(t_k, x) := \mathbb{E}\left[\int_{t_k}^{t_{k+1}} f_i\left(s, X_s^{t_k,x}\right) dt + \hat{v}_i\left(t_{k+1}, X_{t_{k+1}}^{t_k,x}\right)\right] , \; k = m-1, \ldots, 0 \tag{2.10}$$

and where the notation $X^{t,x} := \left(X_s^{t,x}\right)_{s \geq t}$ refers to the process X conditioned on the initial value $X_t = x$.

If, moreover, the cost function k is such that at most one switch can occur on a given date t_k (triangular condition), then equation (2.8) can be simplified into:

$$\hat{v}_i(t_k, x) = \max_{j \in \mathbb{I}_q} \left\{E_j(t_k, x) - k(t_k, j - i)\, \mathbf{1}_{\{j \neq i\}}\right\} \tag{2.11}$$

which is explicit in the sense that $\hat{v}_\cdot(t_k, .)$ is expressed as a function of $\hat{v}_\cdot(t_{k+1}, .)$.

In practice, apart from the potential approximation of the stochastic process X and of the final values (2.9), the difficulty lies in the efficient computation of the conditional expectations (2.10).

In the American options literature, various approaches have been developed to solve (2.11) efficiently. Notable examples are the least-squares approach (Longstaff and Schwartz [2001], Tsitsiklis and Van Roy [2001]), the quantization approach and the Malliavin calculus based formulation (see Bouchard and Warin [2011] for a thorough comparison and improvements of these techniques). In the spirit of Carriere [1996], one may also consider non-parametric regression (see Kohler [2010] and Todorović [2007]) combined with speeding up techniques like Kd-trees (Gray and Moore [2003], Lang et al. [2005]) or the Fast Gauss Transform (Yang et al. [2003], Morariu et al. [2009], Raykar et al. [2010], Spivak et al. [2010], Sampath et al. [2010]) in the case of kernel regression.

Here, we intend to solve (2.1) on numerical applications which bears the particularity of handling stochastic processes in high dimension ($\dim(X) = d \gg 3$, with however $\dim(I) = d' \approx 3$, see Section 5). For such problems, the most adequate technique so far seems to be the local regression method developed in Bouchard and Warin [2011]. We are thus going to make use of this specific method to solve (2.11) in practice.

In the following, a detailed analysis of the algorithm we suggested above is provided.

3 Numerical approximation and convergence analysis

This Section is devoted to the precise description of the resolution of (2.1), in the lines of the discussions from Subsection 2.3. Moreover, the convergence rate of the proposed algorithm towards the exact value function v will be precisely assessed.

3.1 Approximations

Recall equation (2.1) defining the value function $v(t,x,i)$:

$$v(t,x,i) = \sup_{\alpha \in \mathcal{A}_{t,i}} \mathbb{E}\left[\int_t^{+\infty} f\left(s, X_s^{t,x}, I_s^{\alpha}\right) ds - \sum_{\tau_n \geq t} k\left(\tau_n, \zeta_n\right)\right] \tag{3.1}$$

We are going to consider the following sequence of approximations.

3.1.1 Finite Horizon

The first step is to reduce set of strategies to a finite horizon:

$$v_T(t,x,i) = \sup_{\alpha \in \mathcal{A}_{t,i}^T} \mathbb{E}\left[\int_t^T f\left(s, X_s^{t,x}, I_s^{\alpha}\right) ds - \sum_{t \leq \tau_n \leq T} k\left(\tau_n, \zeta_n\right) + g_f\left(T, X_T^{t,x}, I_T^{\alpha}\right)\right] \tag{3.2}$$

$$g_f(T,x,i) := \mathbb{E}\left[\int_T^{+\infty} f\left(s, X_s^{T,x}, i\right) ds\right] \tag{3.3}$$

where $0 \leq t \leq T < +\infty$, and $\mathcal{A}_{t,i}^T \subset \mathcal{A}_{t,i}$ is the subset of strategies without switches after time T. Hence the final value g_f corresponds to the remaining gain after T. Alternatively, one may choose, for convenience, another final value g, as long as it satisfies appropriate conditions (cf. equation (3.19)). The difference between the two value functions is quantified in Proposition 3.1. From now on, we choose one such g function and fix it.

3.1.2 Time discretization

Then, we discretize the time segment $[0,T]$. Introduce a time grid $\Pi = \{t_0 = 0 < t_1 < \ldots < t_N = T\}$ with constant mesh h. Consider the following approximation:

$$v_\Pi(t,x,i) = \sup_{\alpha \in \mathcal{A}_{t,i}^\Pi} \mathbb{E}\left[\int_t^T f\left(s, X_s^{t,x}, I_s^{\alpha}\right) ds - \sum_{t \leq \tau_n \leq T} k\left(\tau_n, \zeta_n\right) + g\left(T, X_T^{t,x}, I_T^{\alpha}\right)\right] \tag{3.4}$$

where $\mathcal{A}_{t,i}^\Pi \subset \mathcal{A}_{t,i}^T$ is the subset of strategies such that switches can only occur at dates $\tau_n \in \Pi \cap [t,T]$.

Now, with a slight abuse of notation, we can safely switch from the notation $\alpha = (\tau_n, \zeta_n)_{n \geq 0}$ to the notation $\alpha = (\tau_n, \iota_n)_{n \geq 0}$ (remember Subsection 2.1), replacing the quantity $\sum_{t \leq \tau_n \leq T} k(\tau_n, \zeta_n)$ by $\sum_{t \leq \tau_n \leq T} k\left(\tau_n, I_{\tau_n - h}^{\alpha}, I_{\tau_n}^{\alpha}\right)$ or by $\sum_{t \leq \tau_n \leq T} k(\tau_n, \iota_{n-1}, \iota_n)$, where $k(t,i,j) = k(t, j-i)$. The error between v_T and v_Π is quantified in Proposition 3.2.

Next we also approximate the stochastic process X by its Euler scheme $\bar{X} = \left(\bar{X}_s\right)_{0 \leq s \leq T}$, with dynamics:

$$\begin{aligned} d\bar{X}_s &= b\left(\pi(s), \bar{X}_{\pi(s)}\right) ds + \sigma\left(\pi(s), \bar{X}_{\pi(s)}\right) dW_s\,, \; 0 \leq s \leq T \\ \bar{X}_0 &= x_0 \in \mathbb{R}^d \end{aligned} \tag{3.5}$$

where $\forall s \in [0,T]$, $\pi(s) := \max\{t \in \Pi; t \leq s\}$. The new value function reads:

$$\bar{v}_\Pi(t,x,i) = \sup_{\alpha \in \mathcal{A}_{t,i}^\Pi} \mathbb{E}\left[\int_t^T f\left(\pi(s), \bar{X}_s^{t,x}, I_s^{\alpha}\right) ds - \sum_{t \leq \tau_n \leq T} k\left(\tau_n, \iota_{n-1}, \iota_n\right) + g\left(T, \bar{X}_T^{t,x}, I_T^{\alpha}\right)\right] \tag{3.6}$$

The error between v_Π and $\bar{v}_\Pi$ is computed in Proposition 3.3.

3.1.3 Space localization

Fix $\varepsilon > 0$. For every $t \in [0,T]$, let $\mathcal{D}_t^\varepsilon$ be a bounded convex domain of $\mathbb{R}^d$, and $\mathcal{P}_t^\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$ be the projection on $\mathcal{D}_t^\varepsilon$. This domain is defined such that:

$$\mathbb{E}\left[\left|\bar{X}_t - \mathcal{P}_t^\varepsilon\left(\bar{X}_t\right)\right|\right] \leq \varepsilon \tag{3.7}$$

Denote this projection as $\bar{X}_t^{\mathcal{D}^\varepsilon}$:

$$\bar{X}_t^{\mathcal{D}^\varepsilon} := \mathcal{P}_\varepsilon\left(\bar{X}_t\right)$$

In other words, $\bar{X}_t^{\mathcal{D}^\varepsilon}$ is equal to $\bar{X}_t$ most of the time (ie. when $\bar{X}_t \in \mathcal{D}_t^\varepsilon$), except when $\bar{X}_t$ is outside $\mathcal{D}_t^\epsilon$, in which case $\bar{X}_t^{\mathcal{D}^\varepsilon}$ corresponds to the projection of $\bar{X}_t$ onto $\mathcal{D}_t^\epsilon$.

From this definition, there exists a constant $C(T,\varepsilon) > 0$ such that $\left|\bar{X}_t^{\mathcal{D}^\varepsilon}\right| \leq C(T,\varepsilon)$ a.s. for every $t \in [0,T]$. In other words, the domain $\mathcal{D}^\varepsilon := \cup_{t\in[0,T]}\mathcal{D}_t^\varepsilon \subset \mathbb{R}^d$ is bounded: $\forall x \in \mathcal{D}^\varepsilon$, $|x| \leq C(T,\varepsilon)$.

Define $\bar{v}_\Pi^{\mathcal{D}^\varepsilon}$ like the value function $\bar{v}_\Pi$ from equation (3.6) but with $\bar{X}$ replaced by $\bar{X}^{\mathcal{D}^\varepsilon}$. The error between those two value function is computed in Proposition 3.4.

To shorten the notations, define the product space $\Pi \times \mathcal{D}_\Pi^\varepsilon$ such that $(t,x) \in \Pi \times \mathcal{D}_\Pi^\varepsilon$ means $t \in \Pi$ and $x \in \mathcal{D}_t^\varepsilon$.

3.1.4 Conditional expectation approximation

From now on, in order to prevent the notations from becoming too cumbersome and clumsy, we are going to skip the $\mathcal{D}^\varepsilon$ index in the following, ie. $\bar{X}_t$ will stand for $\bar{X}_t^{\mathcal{D}^\varepsilon}$, and $\bar{v}_\Pi$ for $\bar{v}_\Pi^{\mathcal{D}^\varepsilon}$.

For the fully discretized problem (3.6), the dynamic programming principle (2.11) becomes:

$$\begin{aligned}
\bar{v}_\Pi(T,x,i) &= g(T,x,i) \\
\bar{v}_\Pi(t_n,x,i) &= \max_{j\in\mathbb{I}_q}\left\{hf(t_n,x,j) - k(t_n,i,j) + \mathbb{E}\left[\bar{v}_\Pi\left(t_{n+1},\bar{X}_{t_{n+1}}^{t_n,x},j\right)\right]\right\},\ n = N-1,\dots,0
\end{aligned} \tag{3.8}$$

The last step is to approximate the conditional expectation appearing in equation (3.8). As discussed in Subsection 2.3, we choose to approximate it using regression. Consider basis functions $(e_k(x))_{1\leq k\leq K}$, $K \in \mathbb{N}\cup\{+\infty\}$, $x \in \mathbb{R}^d$. For suitable functions $\varphi : \Pi \times \mathbb{R}^d \times \mathbb{I}_q \to \mathbb{R}$, define:

$$\tilde{\lambda} = \tilde{\lambda}_i^{t_n}(\varphi) := \arg\min_{\lambda\in\mathbb{R}^K} \mathbb{E}\left[\left(\varphi\left(t_{n+1},\bar{X}_{t_{n+1}},i\right) - \sum_{k=1}^K \lambda_k e_k\left(\bar{X}_{t_n}\right)\right)^2\right] \tag{3.9}$$

Moreover (see Bouchard and Touzi [2004], Tan [2011]), suppose that there exists known bounds $\underline{\Gamma}^{t_n,x}(\varphi)$ and $\overline{\Gamma}^{t_n,x}(\varphi)$ on $\mathbb{E}\left[\varphi\left(t_{n+1},\bar{X}_{t_{n+1}}^{t_n,x},i\right)\right]$:

$$\underline{\Gamma}^{t_n,x}(\varphi) \leq \mathbb{E}\left[\varphi\left(t_{n+1},\bar{X}_{t_{n+1}}^{t_n,x},i\right)\right] \leq \overline{\Gamma}^{t_n,x}(\varphi) \tag{3.10}$$

Then, the quantity $\mathbb{E}\left[\varphi\left(t_{n+1},\bar{X}_{t_{n+1}}^{t_n,x},i\right)\right]$,$i \in \mathbb{I}_q$ is approximated by:

$$\tilde{\mathbb{E}}\left[\varphi\left(t_{n+1},\bar{X}_{t_{n+1}}^{t_n,x},i\right)\right] := \underline{\Gamma}^{t_n,x}(\varphi) \vee \sum_{k=1}^K \tilde{\lambda}_k e_k(x) \wedge \overline{\Gamma}^{t_n,x}(\varphi) \tag{3.11}$$

and is used to define the next approximation $\tilde{v}_\Pi$ of the value function:

$$\begin{aligned}
\tilde{v}_\Pi(T,x,i) &= g(T,x,i) \\
\tilde{v}_\Pi(t_n,x,i) &= \max_{j\in\mathbb{I}_q}\left\{hf(t_n,x,j) - k(t_n,i,j) + \tilde{\mathbb{E}}\left[\tilde{v}_\Pi\left(t_{n+1},\bar{X}_{t_{n+1}}^{t_n,x},j\right)\right]\right\},\ n = N-1,\dots,0
\end{aligned} \tag{3.12}$$

Interesting discussions on the choice of function basis are provided in Bouchard and Warin [2011]. In particular they propose the use of local polynomials, which is numerically efficient and well-suited

for our large-dimensional problems. However, in this section of convergence assessments, we will, for the sake of simplicity, restrict our study to a function basis of local hypercubes (as in Tan [2011] and Gobet et al. [2005]). The error between $\bar{v}_\Pi$ and $\tilde{v}_\Pi$ is computed in Proposition 3.6.

Finally, let $\left(\bar{X}_{t_n}^m\right)_{1\leq n\leq N}^{1\leq m\leq M}$ be a finite sample of size M of paths of the process $\bar{X}$. The final step is to replace the regression (3.9) by a regression on this sample:

$$\hat{\lambda} = \hat{\lambda}_i^{t_n}(\varphi) := \arg\min_{\lambda\in\mathbb{R}^K} \frac{1}{M}\sum_{m=1}^{M}\left[\left(\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^m, i\right) - \sum_{k=1}^{K}\lambda_k e_k\left(\bar{X}_{t_n}^m\right)\right)^2\right] \tag{3.13}$$

Then the quantity $\mathbb{E}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x}, i\right)\right]$,$i \in \mathbb{I}_q$ is approximated by:

$$\hat{\mathbb{E}}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x}, i\right)\right] := \underline{\Gamma}^{t_n,x}(\varphi) \vee \sum_{k=1}^{K}\hat{\lambda}_k e_k(x) \wedge \overline{\Gamma}^{t_n,x}(\varphi) \tag{3.14}$$

leading to the final, computable approximation $\hat{v}_\Pi$ of the value function:

$$\begin{aligned}\hat{v}_\Pi(T,x,i) &= g(T,x,i)\\ \hat{v}_\Pi(t_n,x,i) &= \max_{j\in\mathbb{I}_q}\left\{hf(t_n,x,j) - k(t_n,i,j) + \hat{\mathbb{E}}\left[\hat{v}_\Pi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x}, j\right)\right]\right\},\ n = N-1,\ldots,1\end{aligned} \tag{3.15}$$

The error between $\tilde{v}_\Pi$ and $\hat{v}_\Pi$ (in fact directy between$\bar{v}_\Pi$ and $\hat{v}_\Pi$) is proven in Proposition 3.7 (on a subset of $\mathbb{R}^d$). Combining all these results, the rate of convergence of $\hat{v}_\Pi$ towards v is summed up in Theorem 3.1.

3.2 Convergence analysis

From now on, we suppose that all the assumptions from Subsection 2.2 are enforced.

3.2.1 Finite Time Horizon

First, we introduce the following notations:

$$H(t,T,x,\alpha) := \int_t^T f\left(s, X_s^{t,x}, I_s^\alpha\right)ds - \sum_{t<\tau_n\leq T} k(\tau_n,\zeta_n) \tag{3.16}$$

$$J(t,T,x,\alpha) := \mathbb{E}\left[H(t,T,x,\alpha)\right] \tag{3.17}$$

for any admissible strategy $\alpha \in \mathcal{A}_{t,i}$. In particular:

$$\begin{aligned}v(t,x,i) &= \sup_{\alpha\in\mathcal{A}_{t,i}} J(t,+\infty,x,\alpha)\\ v_T(t,x,i) &= \sup_{\alpha\in\mathcal{A}_{t,i}^T} J(t,+\infty,x,\alpha)\end{aligned} \tag{3.18}$$

Lemma 3.1. *There exists $C > 0$ such that $\forall (t,x,i) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^{d'}$:*

$$0 \leq v(t,x,i) - v_T(t,x,i) \leq C(1+|x|)e^{-\bar{\rho}t\vee T - \rho_1 t}$$

Proof. Fix $(t,x,i) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^{d'}$. First, using equation (3.18):

$$v_T(t,x,i) = \sup_{\alpha\in\mathcal{A}_{t,i}^T} J(t,+\infty,x,\alpha) \leq \sup_{\alpha\in\mathcal{A}_{t,i}} J(t,+\infty,x,\alpha) = v(t,x,i)$$

which provides the first inequality. Consider now the second inequality. Choose $\varepsilon > 0$. From the definition of v (equation (3.1)) there exists a strategy $\alpha^\varepsilon \in \mathcal{A}_{t,i}$ such that:

$$v(t,x,i) - \varepsilon \leq J(t,+\infty,x,\alpha^\varepsilon) \leq v(t,x,i)$$

Define the truncated strategy $\alpha_T^\varepsilon \in \mathcal{A}_{t,i}^T$ such that $\forall s \in [t,T]$, $I_s^{\alpha_T^\varepsilon} = I_s^{\alpha^\varepsilon}$ and $\forall s > T$, $I_s^{\alpha_T^\varepsilon} = I_T^{\alpha^\varepsilon}$. In order not to mix up the variables τ_n and ζ_n from different strategies, we add the name of the strategy in index when needed. Then:

$$\begin{aligned}
& H\left(t,+\infty,x,\alpha^\varepsilon\right) - H\left(t,+\infty,x,\alpha_T^\varepsilon\right) \\
= & \left\{\int_t^{+\infty} f\left(s, X_s^{t,x}, I_s^{\alpha^\varepsilon}\right) ds - \sum_{\tau_n^{\alpha^\epsilon} \geq t} k\left(\tau_n^{\alpha^\epsilon}, \zeta_n^{\alpha^\epsilon}\right)\right\} \\
& - \left\{\int_t^{+\infty} f\left(s, X_s^{t,x}, I_s^{\alpha_T^\varepsilon}\right) ds - \sum_{\tau_n^{\alpha_T^\epsilon} \geq t} k\left(\tau_n^{\alpha_T^\epsilon}, \zeta_n^{\alpha_T^\epsilon}\right)\right\} \\
= & \left\{\int_t^{+\infty} f\left(s, X_s^{t,x}, I_s^{\alpha^\varepsilon}\right) ds - \sum_{\tau_n^{\alpha^\epsilon} \geq t} k\left(\tau_n^{\alpha^\epsilon}, \zeta_n^{\alpha^\epsilon}\right)\right\} \\
& - \left\{\int_t^{t\vee T} f\left(s, X_s^{t,x}, I_s^{\alpha^\varepsilon}\right) ds + \int_{t\vee T}^{+\infty} f\left(s, X_s^{t,x}, I_{t\vee T}^{\alpha^\varepsilon}\right) ds - \sum_{t\vee T \geq \tau_n^{\alpha^\epsilon} \geq t} k\left(\tau_n^{\alpha^\epsilon}, \zeta_n^{\alpha^\epsilon}\right)\right\} \\
= & \int_{t\vee T}^{+\infty} f\left(s, X_s^{t,x}, I_s^{\alpha^\varepsilon}\right) ds - \int_{t\vee T}^{+\infty} f\left(s, X_s^{t,x}, I_{t\vee T}^{\alpha^\varepsilon}\right) ds - \sum_{\tau_n^{\alpha^\epsilon} \geq t\vee T} k\left(\tau_n^{\alpha^\epsilon}, \zeta_n^{\alpha^\epsilon}\right) \\
\leq & \int_{t\vee T}^{+\infty} f\left(s, X_s^{t,x}, I_s^{\alpha^\varepsilon}\right) ds - \int_{t\vee T}^{+\infty} f\left(s, X_s^{t,x}, I_{t\vee T}^{\alpha^\varepsilon}\right) ds
\end{aligned}$$

as $k(s,0) = 0$ and $k \geq 0$ (Assumption 5). Hence, using Jensen's inequality and equation (2.4), $\exists C > 0$ such that:

$$\begin{aligned}
& \left|J\left(t,+\infty,x,\alpha^\varepsilon\right) - J\left(t,+\infty,x,\alpha_T^\varepsilon\right)\right| \\
\leq & \mathbb{E}\left[\left|H\left(t,+\infty,x,\alpha^\varepsilon\right) - H\left(t,+\infty,x,\alpha_T^\varepsilon\right)\right|\right] \\
\leq & \mathbb{E}\left[\int_{t\vee T}^{+\infty} \left|f\left(s, X_s^{t,x}, I_s^{\alpha^\varepsilon}\right)\right| ds\right] + \mathbb{E}\left[\int_{t\vee T}^{+\infty} \left|f\left(s, X_s^{t,x}, I_{t\vee T}^{\alpha^\varepsilon}\right)\right| ds\right] \\
\leq & C\left(1+|x|\right) e^{-\bar{\rho} t\vee T - \rho_1 t}
\end{aligned}$$

Finally, given that:

$$\begin{aligned}
v\left(t,x,i\right) & \leq \varepsilon + J\left(t,+\infty,x,\alpha^\varepsilon\right) \\
v_T\left(t,x,i\right) & \geq J\left(t,+\infty,x,\alpha_T^\varepsilon\right)
\end{aligned}$$

the following holds:

$$\begin{aligned}
v\left(t,x,i\right) - v_T\left(t,x,i\right) & \leq \varepsilon + J\left(t,+\infty,x,\alpha^\varepsilon\right) - J\left(t,+\infty,x,\alpha_T^\varepsilon\right) \\
& \leq \varepsilon + C\left(1+|x|\right) e^{-\bar{\rho} t\vee T - \rho_1 t}
\end{aligned}$$

As this is true for any $\varepsilon > 0$, and as C, ρ and ρ_1 do not depend on ε, the proposition is proved. □

Now, we focus on the final boundary g_f. For the time being, denote the value function (3.2) as $v_T^{g_f}$ to emphasis the choice of final values. As a consequence of equation (2.4), $\forall (x,i) \in \mathbb{R}^d \times \mathbb{I}_q$:

$$\left|g_f(T,x,i)\right| \leq C\left(1+|x|\right) e^{-\rho T} \tag{3.19}$$

Hence, define the class Θ_{g_f} of Lipschitz functions from $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{I}_q$ into $\mathbb{R}$ such that $\forall g \in \Theta_{g_f}$, $\forall (T,x,x',i) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{I}_q$:

$$\begin{aligned}
\left|g(T,x,i) - g(T,x',i)\right| & \leq C_g e^{-\rho T}\left|x - x'\right| & (3.20) \\
\left|g(T,x,i)\right| & \leq C_g e^{-\rho T}\left(1+|x|\right) & (3.21)
\end{aligned}$$

for some $C_g > 0$. Obviously $g_f \in \Theta_{g_f}$. Now, for any $g \in \Theta_{g_f}$, denote as v_T^g the value function defined as in equation (3.2) except for the final value, taken as g instead of g_f. We are going to show that

the precise approximation error due to the choice of final value g does not matter much as long as g is chosen in this class Θ_{g_f}.

Lemma 3.2. *There exists $C > 0$ such that $\forall\,(t,x,i) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{I}_q$:*

$$\left|v_T^{g_f}(t,x,i) - v_T^{g}(t,x,i)\right| \leq C\,(1+|x|)\,e^{-\bar{\rho}t \vee T - \rho_1 t}$$

Proof. Fix $(t,x,i) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{I}_q$. To shorten the proof, we assume that $v_T^{g_f}$ (resp. v_T^{g}) admits an optimal strategy $\alpha_f^* \in \mathcal{A}_{t,i}^T$ (resp. $\alpha^* \in \mathcal{A}_{t,i}^T$) (this assumption can then be relaxed using ε-optimal strategies as in the proof of Proposition 3.1). Therefore:

$$\begin{aligned}
v_T^{g_f}(t,x,i) - v_T^{g}(t,x,i) &= J\left(t,T,x,\alpha_f^*\right) + \mathbb{E}\left[g_f\left(T, X_T^{t,x}, I_T^{\alpha_f^*}\right)\right] - J\left(t,T,x,\alpha^*\right) - \mathbb{E}\left[g\left(T, X_T^{t,x}, I_T^{\alpha^*}\right)\right] \\
&= J\left(t,T,x,\alpha_f^*\right) + \mathbb{E}\left[g\left(T, X_T^{t,x}, I_T^{\alpha_f^*}\right)\right] - J\left(t,T,x,\alpha^*\right) - \mathbb{E}\left[g\left(T, X_T^{t,x}, I_T^{\alpha^*}\right)\right] \\
&\quad + \mathbb{E}\left[g_f\left(T, X_T^{t,x}, I_T^{\alpha_f^*}\right) - g\left(T, X_T^{t,x}, I_T^{\alpha_f^*}\right)\right] \\
&\leq \mathbb{E}\left[g_f\left(T, X_T^{t,x}, I_T^{\alpha_f^*}\right) - g\left(T, X_T^{t,x}, I_T^{\alpha^*}\right)\right] \\
&\leq C\left(1 + \mathbb{E}\left[\left|X_T^{t,x}\right|\right]\right) e^{-\rho T} \\
&\leq C\,(1+|x|)\,e^{-\bar{\rho}t \vee T - \rho_1 t}
\end{aligned}$$

Symmetrically, the same inequality holds for $v_T^{g}(t,x,i) - v_T^{g_f}(t,x,i)$, ending the proof. □

Remark 3.1. Under the assumptions from Subsection 2.2, one may use Theorem 3.1 from Hu and Tang [2010] to prove the existence and uniqueness of a unique optimal strategy α^* for the value function (3.2), satisfying:

$$\mathbb{E}\left[\left|\sum_{0 \leq \tau_n^{\alpha^*} \leq T} k\left(\tau_n^{\alpha^*}, \zeta_n^{\alpha^*}\right)\right|^2\right] < \infty$$

Proposition 3.1. *There exists $C > 0$ such that $\forall\,(t,x,i) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{I}_q$ and $\forall g \in \Theta_{g_f}$:*

$$\left|v(t,x,i) - v_T^{g}(t,x,i)\right| \leq C\,(1+|x|)\,e^{-\bar{\rho}t \vee T - \rho_1 t}$$

Proof. Combine Lemmas 3.1 and 3.2. □

From now on, we choose and keep one final value function $g \in \Theta_{g_f}$, and remove the index g from the notation of v and its subsequent approximations.

3.2.2 Time Discretization

Proposition 3.2. *There exists a positive constant C such that for any $(t,x,i) \in \Pi \times \mathbb{R}^d \times \mathbb{I}_q$:*

$$\left|v_T(t,x,i) - v_\Pi(t,x,i)\right| \leq C e^{-\rho t}\left(1 + |x|^{\frac{3}{2}}\right) h^{\frac{1}{2}} \tag{3.22}$$

Proof. Under the assumptions from Subsection 2.2, one can apply Theorem 3.1 from Gassiat et al. [2012] to prove (3.22), noticing that the cost function k does not depend on the state variable x. Use the discounting factor in the definition of f to factor the $e^{-\rho t}$ term and to get that C does not depend on T. Another alternative to get this rate of $h^{\frac{1}{2}}$ is to work with the reflected BSDE representation of v_Π, as in Carmona and Ludkovski [2008] (adapting Bouchard and Touzi [2004]) or Chassagneux et al. [2011]. □

Remark 3.2. Were the cost function k to depend on the state variable, the upper bound in Proposition 3.2 would only be $Ce^{-\rho t}\left(1 + |x|^{\frac{5}{2}}\right)\left(h \log\left(\frac{2T}{h}\right)\right)^{\frac{1}{2}}$, as stated in Gassiat et al. [2012] (making use of results from Fischer and Nappo [2009]).

Now that T and g are fixed, we can define, in the spirit of equations (3.16) and (3.17), the following quantities:

$$H(t,x,\alpha) := \int_t^T f\left(s, X_s^{t,x}, I_s^\alpha\right) ds - \sum_{t\le\tau_n\le T} k\left(\tau_n, \iota_{n-1}, \iota_n\right) + g\left(t\vee T, X_{t\vee T}^{t,x}, I_{t\vee T}^\alpha\right) \quad (3.23)$$

$$J(t,x,\alpha) := \mathbb{E}\left[H(t,x,\alpha)\right] \quad (3.24)$$

$$\bar{H}(t,x,\alpha) := \int_t^T f\left(\pi(s), \bar{X}_s^{t,x}, I_s^\alpha\right) ds - \sum_{t\le\tau_n\le T} k\left(\tau_n, \iota_{n-1}, \iota_n\right) + g\left(t\vee T, \bar{X}_{t\vee T}^{t,x}, I_{t\vee T}^\alpha\right) \quad (3.25)$$

$$\bar{J}(t,x,\alpha) := \mathbb{E}\left[\bar{H}(t,x,\alpha)\right] \quad (3.26)$$

for any admissible strategy $\alpha \in \mathcal{A}_{t,i}^\Pi$.

Proposition 3.3. *There exists $C>0$ such that for any $(t,x,i)\in\Pi\times\mathbb{R}^d\times\mathbb{I}_q$:*

$$\left|v_\Pi(t,x,i) - \bar{v}_\Pi(t,x,i)\right| \le Ce^{-\rho t}h^{\frac{1}{2}}$$

Proof. For these discretized problems, the existence of optimal controls α^* and $\bar{\alpha}^*$ is granted. Hence:

$$\begin{aligned}
v_\Pi(t,x,i) - \bar{v}_\Pi(t,x,i) &= J(t,x,\alpha^*) - \bar{J}(t,x,\bar{\alpha}^*) \\
&= J(t,x,\alpha^*) - \bar{J}(t,x,\alpha^*) + \left\{\bar{J}(t,x,\alpha^*) - \bar{J}(t,x,\bar{\alpha}^*)\right\} \\
&\le J(t,x,\alpha^*) - \bar{J}(t,x,\alpha^*) \\
&= \int_t^T e^{-\rho s}\mathbb{E}\left[\tilde{f}\left(s, X_s^{t,x,I_s^{\alpha^*}}\right) - \tilde{f}\left(\pi(s), \bar{X}_s^{t,x,I_s^{\alpha^*}}\right)\right] ds \\
&\quad + \mathbb{E}\left[g\left(T, X_T^{t,x,I_T^{\alpha^*}}\right) - g\left(T, \bar{X}_T^{t,x,I_T^{\alpha^*}}\right)\right] \\
&\le C_f\int_t^T e^{-\rho s}\mathbb{E}\left[\left|X_s^{t,x} - \bar{X}_s^{t,x}\right|\right] ds + C_g e^{-\rho T}\mathbb{E}\left[\left|X_T^{t,x} - \bar{X}_T^{t,x}\right|\right] \\
&\le Ce^{-\rho t}\mathbb{E}\left[\sup_{t\le s\le T}\left|X_s^{t,x} - \bar{X}_s^{t,x}\right|\right] \\
&\le Ce^{-\rho t}h^{\frac{1}{2}}
\end{aligned}$$

using the strong convergence speed of the Euler scheme on $[t,T]$. Symmetrically, the same inequality holds for $\bar{v}_\Pi(t,x,i) - v_\Pi(t,x,i)$, ending the proof. □

Finally, below are some bounds on v_Π and $\bar{v}_\Pi$, which will be useful later:

$$\left|v_\Pi(t,x,i)\right| \le Ce^{-\rho t}(1+|x|) \quad (3.27)$$

$$\left|\bar{v}_\Pi(t,x,i)\right| \le Ce^{-\rho t}\left(1+|x|+h^{\frac{1}{2}}\right) \quad (3.28)$$

For the first inequality, proceed as with equation (2.5). For the second, combine equation (3.27) with Proposition 3.3.

3.2.3 Space localization

Recall from Paragraph 3.1.3 the definition of the bounded domain $\mathcal{D}^\varepsilon$.

Proposition 3.4. *There exists $C>0$ such that for any $(x,i)\in\mathbb{R}\times\mathbb{I}_q$:*

$$\left|\bar{v}_\Pi(0,x,i) - \bar{v}_\Pi^{\mathcal{D}^\varepsilon}(0,x,i)\right| \le C\varepsilon$$

where ε is defined in equation (3.7).

Proof. Recall the definitions of $\bar{H}(t,x,\alpha)$ (equation (3.25)) and $\bar{J}(t,x,\alpha)$ (equation (3.26)), and define the quantities $\bar{H}^{\mathcal{D}^\varepsilon}(t,x,\alpha)$ and $\bar{J}^{\mathcal{D}^\varepsilon}(t,x,\alpha)$ like $\bar{H}(t,x,\alpha)$ and $\bar{J}(t,x,\alpha)$ but with $\bar{X}$ replaced by $\bar{X}^{\mathcal{D}^\varepsilon}$. Then, for every $(t,x,i) \in \Pi \times \mathbb{R}^d \times \mathbb{I}_q$ and $\alpha \in \mathcal{A}_{t,i}^{\Pi}$:

$$\begin{aligned}\bar{J}(t,x,\alpha) &= \bar{J}^{\mathcal{D}^\varepsilon}(t,x,\alpha) + \\ &\int_t^T \mathbb{E}\left[f\left(\pi(s), \bar{X}_s^{t,x}, I_s^\alpha\right) - f\left(\pi(s), \bar{X}_s^{\mathcal{D}^\varepsilon,t,x}, I_s^\alpha\right)\right] ds \\ &+\mathbb{E}\left[g\left(T, \bar{X}_T^{t,x}, I_T^\alpha\right) - g\left(T, \bar{X}_T^{\mathcal{D}^\varepsilon,t,x}, I_T^\alpha\right)\right]\end{aligned}$$

But:

$$\begin{aligned}&\left|\int_t^T \mathbb{E}\left[f\left(\pi(s), \bar{X}_s^{t,x}, I_s^\alpha\right) - f\left(\pi(s), \bar{X}_s^{\mathcal{D}^\varepsilon,t,x}, I_s^\alpha\right)\right] ds + \mathbb{E}\left[g\left(T, \bar{X}_T^{t,x}, I_T^\alpha\right) - g\left(T, \bar{X}_T^{\mathcal{D}^\varepsilon,t,x}, I_T^\alpha\right)\right]\right| \\ &\leq C_f \int_t^T e^{-\rho s}\mathbb{E}\left[\left|\bar{X}_s^{t,x} - \bar{X}_s^{\mathcal{D}^\varepsilon,t,x}\right|\right] ds + C_g e^{-\rho T}\mathbb{E}\left[\left|\bar{X}_T^{t,x} - \bar{X}_T^{\mathcal{D}^\varepsilon,t,x}\right|\right]\end{aligned}$$

It follows that:

$$\left|\bar{v}_\Pi(t,x,i) - \bar{v}_\Pi^{\mathcal{D}^\varepsilon}(t,x,i)\right| \leq C_f \int_t^T e^{-\rho s}\mathbb{E}\left[\left|\bar{X}_s^{t,x} - \bar{X}_s^{\mathcal{D}^\varepsilon,t,x}\right|\right] ds + C_g e^{-\rho T}\mathbb{E}\left[\left|\bar{X}_T^{t,x} - \bar{X}_T^{\mathcal{D}^\varepsilon,t,x}\right|\right]$$

In particular, at $t=0$, using equation (3.7), $\exists C > 0$ such that:

$$\left|\bar{v}_\Pi(0,x,i) - \bar{v}_\Pi^{\mathcal{D}^\varepsilon}(0,x,i)\right| \leq C\varepsilon$$

□

3.2.4 Conditional expectation approximation

Now the domain $\mathcal{D}^\varepsilon$ is chosen and fixed, and from now on, with a slight abuse of notation, we will omit it from the notations.

For every $t_n \in \Pi$, consider a partition of $\mathcal{D}_{t_n}^\varepsilon$ into hypercubes $\left(B_{t_n}^k\right)_{k=1,\dots,K}$, i.e., $\cup_{k=1,\dots,K} B_{t_n}^k = \mathcal{D}_{t_n}^\varepsilon$ and $B_{t_n}^i \cap B_{t_n}^j = \emptyset \ \forall i \neq j$. The lengths of the edges of the hypercubes, in each dimension, are supposed smaller than a given constant $\delta > 0$ (in particular, the volume of each hypercube $B_{t_n}^k$ is smaller than δ^d). The fact that the partition can vary over time enables to encompass the kind of adaptative partitions described in Bouchard and Warin [2011].

Remark 3.3. With this choice of basis, $\tilde{\lambda}_i^{t_n}(\varphi)$ (equation (3.9)) and $\hat{\lambda}_i^{t_n}(\varphi)$ (equation (3.13)) become:

$$\begin{aligned}\tilde{\lambda}_i^{t_n}(\varphi) &= \frac{\mathbb{E}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}, i\right)\mathbf{1}\left\{\bar{X}_{t_n} \in B_{t_n}^k\right\}\right]}{\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}^k\right)} = \mathbb{E}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}, i\right) \middle| \bar{X}_{t_n} \in B_{t_n}^k\right],\ 1 \leq k \leq K \\ \hat{\lambda}_i^{t_n}(\varphi) &= \frac{\frac{1}{M}\sum_{m=1}^M \varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^m, i\right)\mathbf{1}\left\{\bar{X}_{t_n}^m \in B_{t_n}^k\right\}}{\frac{1}{M}\sum_{m=1}^M \mathbf{1}\left\{\bar{X}_{t_n}^m \in B_{t_n}^k\right\}},\ 1 \leq k \leq K\end{aligned}$$

Definition 3.1. For every $x \in \mathcal{D}_{t_n}^\varepsilon$, define $B_{t_n}(x)$ as the unique hypercube in the partition which contains x at time t_n.

Definition 3.2. Using Definition 3.1 and Remark 3.3, $\forall (t_n, x, i) \in \Pi \times \mathcal{D}_\Pi^\varepsilon \times \mathbb{I}_q$ define:

$$\begin{aligned}\tilde{\lambda}_i^{t_n,x}(\varphi) &= \frac{\mathbb{E}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}, i\right)\mathbf{1}\left\{\bar{X}_{t_n} \in B_{t_n}(x)\right\}\right]}{\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)} = \mathbb{E}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}, i\right) \middle| \bar{X}_{t_n} \in B_{t_n}(x)\right] \\ \hat{\lambda}_i^{t_n,x}(\varphi) &= \frac{\frac{1}{M}\sum_{m=1}^M \varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^m, i\right)\mathbf{1}\left\{\bar{X}_{t_n}^m \in B_{t_n}(x)\right\}}{\frac{1}{M}\sum_{m=1}^M \mathbf{1}\left\{\bar{X}_{t_n}^m \in B_{t_n}(x)\right\}}\end{aligned}$$

Definition 3.3. Recalling equations (3.11) and (3.14) and using Definition 3.2, define, for any $(t_n, x, j) \in \Pi \times \mathcal{D}_{\Pi}^{\varepsilon} \times \mathbb{I}_q$ and any measurable function $\varphi : \Pi \times \mathbb{R}^d \times \mathbb{I}_q \to \mathbb{R}$, the following quantities:

$$\begin{aligned}
\Phi_j^{t_n,x}(\varphi) &:= \mathbb{E}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x}, j\right)\right] \\
\tilde{\Phi}_j^{t_n,x}(\varphi) &:= \tilde{\mathbb{E}}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x}, j\right)\right] = \underline{\Gamma}^{t_n,x}(\varphi) \vee \tilde{\lambda}_j^{t_n,x}(\varphi) \wedge \overline{\Gamma}^{t_n,x}(\varphi) \\
\hat{\Phi}_j^{t_n,x}(\varphi) &:= \hat{\mathbb{E}}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x}, j\right)\right] = \underline{\Gamma}^{t_n,x}(\varphi) \vee \hat{\lambda}_j^{t_n,x}(\varphi) \wedge \overline{\Gamma}^{t_n,x}(\varphi)
\end{aligned}$$

where (recalling equation 3.10) $\underline{\Gamma}^{t_n,x}(\varphi)$ and $\overline{\Gamma}^{t_n,x}(\varphi)$ are lower and upper bounds on $\Phi_j^{t_n,x}(\varphi)$:

$$\underline{\Gamma}^{t_n,x}(\varphi) \leq \Phi_j^{t_n,x}(\varphi) \leq \overline{\Gamma}^{t_n,x}(\varphi)$$

Remark 3.4. Using Definition 3.3, the dynamic programming equations (3.8), (3.12) and (3.15) become:

$$\begin{aligned}
\bar{v}_{\Pi}(T, x, i) &= g(T, x, i) \\
\bar{v}_{\Pi}(t_n, x, i) &= \max_{j \in \mathbb{I}_q} \left\{ hf(t_n, x, j) - k(t_n, i, j) + \Phi_j^{t_n,x}(\bar{v}_{\Pi}) \right\}, \ n = N-1, \ldots, 0 \\
\\
\tilde{v}_{\Pi}(T, x, i) &= g(T, x, i) \\
\tilde{v}_{\Pi}(t_n, x, i) &= \max_{j \in \mathbb{I}_q} \left\{ hf(t_n, x, j) - k(t_n, i, j) + \tilde{\Phi}_j^{t_n,x}(\tilde{v}_{\Pi}) \right\}, \ n = N-1, \ldots, 0 \\
\\
\hat{v}_{\Pi}(T, x, i) &= g(T, x, i) \\
\hat{v}_{\Pi}(t_n, x, i) &= \max_{j \in \mathbb{I}_q} \left\{ hf(t_n, x, j) - k(t_n, i, j) + \hat{\Phi}_j^{t_n,x}(\hat{v}_{\Pi}) \right\}, \ n = N-1, \ldots, 0
\end{aligned}$$

In order to complete Remark 3.4, we compute below possible bounding functions $\underline{\Gamma}^{t_n,x}(\bar{v}_{\Pi})$ and $\overline{\Gamma}^{t_n,x}(\bar{v}_{\Pi})$ for $\Phi_j^{t_n,x}(\bar{v}_{\Pi})$:

Proposition 3.5. *$\exists C > 0$ such that $\forall (t_n, x, j) \in \Pi \times \mathcal{D}_{\Pi}^{\varepsilon} \times \mathbb{I}_q$:*

$$\left|\Phi_j^{t_n,x}(\bar{v}_{\Pi})\right| \leq \Gamma^{t_n,x}(\bar{v}_{\Pi}) := Ce^{-\rho t_n}\left(1 + |x| + \sqrt{h}\right)$$

Proof. Combine equation (3.28) with the dynamic programming equation from Remark 3.4. □

Lemma 3.3. *Consider a measurable function $\varphi : \Pi \times \mathbb{R}^d \times \mathbb{I}_q \to \mathbb{R}$. Suppose that, for a fixed $t_{n+1} \in \Pi$, it is Lipschitz with constant C_{n+1}, uniformly in j: $\forall (x_1, x_2, j) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{I}_q$*

$$|\varphi(t_{n+1}, x_1, j) - \varphi(t_{n+1}, x_2, j)| \leq C_{n+1}|x_1 - x_2| \tag{3.29}$$

Then $\Phi_j^{t_n,x}(\varphi)$ is Lipschitz with constant $C_{n+1}(1 + \theta(h))$, uniformly in j, where:

$$\theta(h) := \sqrt{1 + (2C_b + C_\sigma^2)h + C_b^2 h^2} - 1$$

In particular $\theta(h) \sim \left(C_b + \frac{1}{2}C_\sigma^2\right)h + \frac{1}{2}C_b^2 h^2 = \mathcal{O}(h)$ when $h \to 0$.

Proof. Choose $(t_n, j, x_1, x_2) \in \Pi \times \mathbb{I}_q \times \mathbb{R}^d \times \mathbb{R}^d$. Then:

$$\begin{aligned}
\left|\Phi_j^{t_n,x_1}(\varphi) - \Phi_j^{t_n,x_2}(\varphi)\right| &= \left|\mathbb{E}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x_1}, j\right) - \varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x_2}, j\right)\right]\right| \\
&\leq \left\|\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x_1}, j\right) - \varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x_2}, j\right)\right\|_1 \\
&\leq \left\|\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x_1}, j\right) - \varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x_2}, j\right)\right\|_2
\end{aligned}$$

Now, using equations (3.29) and (3.5), and G denoting a d-dimensional standard Gaussian random variable:

$$\begin{aligned}
& \mathbb{E}\left[\left(\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x_1}, j\right) - \varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x_2}, j\right)\right)^2\right] \\
\leq\ & C_{n+1}^2 \mathbb{E}\left[\left(\bar{X}_{t_{n+1}}^{t_n,x_1} - \bar{X}_{t_{n+1}}^{t_n,x_2}\right)^2\right] \\
\leq\ & C_{n+1}^2 \mathbb{E}\left[\left(x_1 - x_2 + h\left(b\left(t_n, x_1\right) - b\left(t_n, x_2\right)\right) + \sqrt{h}\left(\sigma\left(t_n, x_1\right) - \sigma\left(t_n, x_2\right)\right) G\right)^2\right] \\
=\ & C_{n+1}^2 \left\{\left(x_1 - x_2 + h\left(b\left(t_n, x_1\right) - b\left(t_n, x_2\right)\right)\right)^2 + h\mathbb{E}\left[\left(\left(\sigma\left(t_n, x_1\right) - \sigma\left(t_n, x_2\right)\right) G\right)^2\right]\right\} \\
\leq\ & C_{n+1}^2 \left(x_1 - x_2\right)^2 \left\{1 + \left(2C_b + C_\sigma^2\right) h + C_b^2 h^2\right\}
\end{aligned}$$

Thus:

$$\left|\Phi_j^{t_n,x_1}\left(\varphi\right) - \Phi_j^{t_n,x_2}\left(\varphi\right)\right| \leq C_{n+1}\sqrt{1 + \left(2C_b + C_\sigma^2\right) h + C_b^2 h^2}\left|x_1 - x_2\right|$$

□

Lemma 3.4. *Consider again a function $\varphi : \Pi \times \mathbb{R}^d \times \mathbb{I}_q \to \mathbb{R}$ such that (3.29) holds for a given $t_{n+1} \in \Pi$. Then, $\forall (x, j) \in \mathcal{D}_{t_n}^\varepsilon \times \mathbb{I}_q$:*

$$\left|\tilde{\lambda}_j^{t_n,x}\left(\varphi\right) - \Phi_j^{t_n,x}\left(\varphi\right)\right| \leq C_{n+1}\delta\left(1 + \theta\left(h\right)\right)$$

in particular:

$$\left|\tilde{\Phi}_j^{t_n,x}\left(\varphi\right) - \Phi_j^{t_n,x}\left(\varphi\right)\right| \leq C_{n+1}\delta\left(1 + \theta\left(h\right)\right) \tag{3.30}$$

Proof. Recalling Definitions 3.1, 3.2 and 3.3, simply remark that:

$$\begin{aligned}
\min_{\tilde{x}\in B(x)} \Phi_j^{t_n,\tilde{x}}\left(\varphi\right) &\leq \Phi_j^{t_n,x}\left(\varphi\right) \leq \max_{\tilde{x}\in B(x)} \Phi_j^{t_n,\tilde{x}}\left(\varphi\right) \\
\min_{\tilde{x}\in B(x)} \Phi_j^{t_n,\tilde{x}}\left(\varphi\right) &\leq \tilde{\lambda}_j^{t_n,x}\left(\varphi\right) \leq \max_{\tilde{x}\in B(x)} \Phi_j^{t_n,\tilde{x}}\left(\varphi\right)
\end{aligned}$$

Now, using Lemma 3.3:

$$\begin{aligned}
\left|\tilde{\lambda}_j^{t_n,x}\left(\varphi\right) - \Phi_j^{t_n,x}\left(\varphi\right)\right| &\leq \max_{\tilde{x}\in B(x)} \Phi_j^{t_n,\tilde{x}}\left(\varphi\right) - \min_{\tilde{x}\in B(x)} \Phi_j^{t_n,\tilde{x}}\left(\varphi\right) \\
&\leq C_{n+1}\left(1 + \theta\left(h\right)\right) \max_{(x_1,x_2)\in B(x)^2} \left|x_1 - x_2\right| \\
&\leq C_{n+1}\left(1 + \theta\left(h\right)\right)\delta
\end{aligned}$$

□

Lemma 3.5. $\forall \left(t_n, x_1, x_2, i\right) \in \Pi \times \left(\mathbb{R}^d\right)^2 \times \mathbb{I}_q$:

$$\left|\bar{v}_\Pi\left(t_n, x_1, i\right) - \bar{v}_\Pi\left(t_n, x_2, i\right)\right| \leq C_n\left|x_1 - x_2\right|$$

where:

$$\begin{aligned}
C_N &= e^{-\rho t_N} C_g \\
C_n &= hC_f e^{-\rho t_n} + C_{n+1}\left(1 + \theta\left(h\right)\right),\ n = N-1, \ldots, 0
\end{aligned} \tag{3.31}$$

Proof. Recall Remark 3.4. We prove the Lemma by induction. First, remark that, using hypothesis (3.20), it holds for $n = N$. Now, suppose that it holds for some $(n+1) \in [1, \ldots, N]$. Then, using Lemma 3.3:

$$\begin{aligned}
& \bar{v}_\Pi\left(t_n, x_1, i\right) \\
=\ & \max_{j\in\mathbb{I}_q}\left\{hf\left(t_n, x_1, j\right) - k\left(t_n, i, j\right) + \Phi_j^{t_n,x_1}\left(\bar{v}_\Pi\right)\right\} \\
=\ & \max_{j\in\mathbb{I}_q}\left\{hf\left(t_n, x_2, j\right) - k\left(t_n, i, j\right) + \Phi_j^{t_n,x_2}\left(\bar{v}_\Pi\right) + h\left(f\left(t_n, x_1, j\right) - f\left(t_n, x_2, j\right)\right) + \left(\Phi_j^{t_n,x_1}\left(\bar{v}_\Pi\right) - \Phi_j^{t_n,x_2}\left(\bar{v}_\Pi\right)\right)\right\} \\
\leq\ & \max_{j\in\mathbb{I}_q}\left\{hf\left(t_n, x_2, j\right) - k\left(t_n, i, j\right) + \Phi_j^{t_n,x_2}\left(\bar{v}_\Pi\right) + he^{-\rho t_n} C_f\left|x_1 - x_2\right| + C_{n+1}\left(1 + \theta\left(h\right)\right)\left|x_1 - x_2\right|\right\} \\
=\ & \bar{v}_\Pi\left(t_n, x_2, i\right) + \left(he^{-\rho t_n} C_f + C_{n+1}\left(1 + \theta\left(h\right)\right)\right)\left|x_1 - x_2\right|
\end{aligned}$$

Symmetrically, the same inequality holds for $\bar{v}_\Pi(t_n, x_2, i) - \bar{v}_\Pi(t_n, x_1, i)$, ending the proof of the Lemma. □

Remark 3.5. Using the recurrence relation (3.31), the Lipschitz coefficients C_n can be explicitly computed, as well as their equivalent when $h \to 0$:

$$\begin{aligned} C_n &= hC_f \sum_{k=n}^{N-1} \left(e^{-\rho h}\right)^k (1+\theta(h))^{k-n} + C_N (1+\theta(h))^{N-n} \\ &= hC_f e^{-\rho t_n} \frac{1 - \left[e^{-\rho h}(1+\theta(h))\right]^{N-n}}{1 - e^{-\rho h}(1+\theta(h))} + C_g e^{-\rho T}(1+\theta(h))^{N-n} \\ \sim_{h\to 0} & \quad e^{-\rho t_n}\left(\frac{C_f}{\tilde{\rho}} + \left(C_g - \frac{C_f}{\tilde{\rho}}\right) e^{-\tilde{\rho}(T-t_n)} + \mathcal{O}(h)\right) \leq C e^{-\rho t_n}(1+\mathcal{O}(h)) \end{aligned}$$

where $\tilde{\rho} := \rho - C_b - \frac{1}{2}C_\sigma^2$ is assumed positive (see Assumption 4). Alternatively, one can obtain this bound using the discrete version of Gronwall's inequality.

Proposition 3.6. *$\exists C > 0$ s.t. $\forall (t,x,i) \in \Pi \times \mathbb{R}^d \times \mathbb{I}_q$:*

$$|\bar{v}_\Pi(t,x,i) - \tilde{v}_\Pi(t,x,i)| \leq C\frac{\delta}{h} e^{-\rho t}$$

Proof. For each $t_n \in \Pi$, we look for an upper bound E_n, independent of x and i, of the quantity $|\bar{v}_\Pi(t_n,x,i) - \tilde{v}_\Pi(t_n,x,i)|$. First:

$$|\bar{v}_\Pi(T,x,i) - \tilde{v}_\Pi(T,x,i)| = |g(T,x,i) - g(T,x,i)| = 0$$

Hence $E_N = 0$. Fix now $n \in [0, N-1]$. Using Remark 3.4:

$$\begin{aligned} \tilde{v}_\Pi(t_n,x,i) &= \max_{j\in\mathbb{I}_q} \left\{ hf(t_n,x,j) - k(t_n,i,j) + \tilde{\Phi}_j^{t_n,x}(\tilde{v}_\Pi)\right\} \\ &= \max_{j\in\mathbb{I}_q} \left\{ hf(t_n,x,j) - k(t_n,i,j) + \Phi_j^{t_n,x}(\bar{v}_\Pi) \right. \\ &\quad + \tilde{\Phi}_j^{t_n,x}(\bar{v}_\Pi) - \Phi_j^{t_n,x}(\bar{v}_\Pi) \\ &\quad \left. + \tilde{\Phi}_j^{t_n,x}(\tilde{v}_\Pi) - \tilde{\Phi}_j^{t_n,x}(\bar{v}_\Pi)\right\} \end{aligned}$$

Using Lemmas 3.4 and 3.5, $\tilde{\Phi}_j^{t_n,x}(\bar{v}_\Pi) - \Phi_j^{t_n,x}(\bar{v}_\Pi) \leq C_{n+1}\delta(1+\theta(h))$ where C_{n+1} is the Lipschitz constant of $\bar{v}_\Pi$ at time t_{n+1} (see Remark 3.5). Moreover,

$$\begin{aligned} \tilde{\Phi}_j^{t_n,x}(\tilde{v}_\Pi) - \tilde{\Phi}_j^{t_n,x}(\bar{v}_\Pi) &\leq \mathbb{E}\left[\tilde{v}_\Pi\left(t_{n+1}, \bar{X}_{t_{n+1}}, j\right) - \bar{v}_\Pi\left(t_{n+1}, \bar{X}_{t_{n+1}}, j\right) \middle| X_{t_n} \in B_{t_n}(x)\right] \\ &\leq E_{n+1} \end{aligned}$$

Hence:

$$\tilde{v}_\Pi(t_n,x,i) \leq \bar{v}_\Pi(t_n,x,i) + C_{n+1}\delta(1+\theta(h)) + E_{n+1}$$

Symmetrically, the same inequality holds for $\bar{v}_\Pi(T,x,i) - \tilde{v}_\Pi(t_n,x,i)$, leading to:

$$|\bar{v}_\Pi(t_n,x,i) - \tilde{v}_\Pi(t_n,x,i)| \leq E_n$$

where:

$$\begin{aligned} E_N &= 0 \\ E_n &= C_{n+1}\delta(1+\theta(h)) + E_{n+1} \end{aligned}$$

Consequently, using Remark 3.5:

$$\begin{aligned}
E_n &= \delta\left(1+\theta\left(h\right)\right)\sum_{k=n+1}^{N} C_k \\
&= \delta\left(1+\theta\left(h\right)\right)\left\{\frac{hC_f}{1-e^{-\rho h}\left(1+\theta\left(h\right)\right)}e^{-\rho t_{n+1}}\frac{1-e^{-\rho(T-t_{n+1})}}{1-e^{-\rho h}}\right. \\
&\quad \left.+\left(C_g-\frac{hC_f}{1-e^{-\rho h}\left(1+\theta\left(h\right)\right)}\right)e^{-\rho T}\frac{\left[\left(1+\theta\left(h\right)\right)\right]^{N-n}-1}{\theta\left(h\right)}\right\} \\
\sim_{h\to 0} &\ \frac{\delta}{h}\left(\frac{C_f}{\tilde{\rho}\rho}e^{-\rho t_{n+1}}\left(1-e^{-\rho(T-t_{n+1})}\right)+\left(C_g-\frac{C_f}{\tilde{\rho}}\right)\frac{e^{-\rho t_n}}{C_b+\frac{1}{2}C_\sigma^2}\left(e^{-\tilde{\rho}(T-t_n)}-e^{-\rho(T-t_n)}\right)\right) \\
&\leq C\frac{\delta}{h}e^{-\rho t_n}
\end{aligned}$$

where $C>0$ does not depend on t_n nor T. □

Lemma 3.6. *Consider a function $\varphi:\Pi\times\mathbb{R}^d\times\mathbb{I}_q\to\mathbb{R}$. For any $p\geq 1$, there exists $C_p\geq 0$ such that $\forall\left(t_n,x,j\right)\in\Pi\times\mathcal{D}_\Pi^\varepsilon\times\mathbb{I}_q$:*

$$\left\|\hat{\Phi}_j^{t_n,x}\left(\varphi\right)-\tilde{\Phi}_j^{t_n,x}\left(\varphi\right)\right\|_p\leq\frac{C_p}{\sqrt{M}}\frac{\sqrt{\tilde{\lambda}_j^{t_n,x}\left(\varphi^2\right)}+\Gamma^{t_n,x}\left(\varphi\right)}{\sqrt{\mathbb{P}\left(\bar{X}_{t_n}\in B_{t_n}\left(x\right)\right)}} \tag{3.32}$$

Proof. It is an extension of Lemma 3.8 in Tan [2011] (itself inspired by Theorem 5.1 in Bouchard and Touzi [2004]). Define:

$$\begin{aligned}
\varepsilon_j^{t_n,x}\left(\varphi\right) &:= \frac{1}{M}\sum_{m=1}^{M}\varphi\left(t_{n+1},\bar{X}_{t_{n+1}}^m,i\right)\mathbf{1}\left\{\bar{X}_{t_n}^m\in B_{t_n}\left(x\right)\right\}-\mathbb{E}\left[\varphi\left(t_{n+1},\bar{X}_{t_{n+1}},i\right)\mathbf{1}\left\{\bar{X}_{t_n}\in B_{t_n}\left(x\right)\right\}\right] \\
\varepsilon_j^{t_n,x}\left(1\right) &:= \frac{1}{M}\sum_{m=1}^{M}\mathbf{1}\left\{\bar{X}_{t_n}^m\in B_{t_n}\left(x\right)\right\}-\mathbb{P}\left(\bar{X}_{t_n}\in B_{t_n}\left(x\right)\right)
\end{aligned}$$

Then:

$$\begin{aligned}
&\left|\hat{\Phi}_j^{t_n,x}\left(\varphi\right)-\tilde{\Phi}_j^{t_n,x}\left(\varphi\right)\right| \\
=\ &\left|\hat{\Phi}_j^{t_n,x}\left(\varphi\right)-\tilde{\Phi}_j^{t_n,x}\left(\varphi\right)\right|\wedge 2\Gamma^{t_n,x}\left(\varphi\right) \\
\leq\ &\left|\hat{\Phi}_j^{t_n,x}\left(\varphi\right)-\tilde{\Phi}_j^{t_n,x}\left(\varphi\right)\right|\mathbf{1}\left\{\frac{\left|\varepsilon_j^{t_n,x}\left(1\right)\right|}{\mathbb{P}\left(\bar{X}_{t_n}\in B_{t_n}\left(x\right)\right)}\leq\frac{1}{2}\right\}+2\Gamma^{t_n,x}\left(\varphi\right)\mathbf{1}\left\{\frac{\left|\varepsilon_j^{t_n,x}\left(1\right)\right|}{\mathbb{P}\left(\bar{X}_{t_n}\in B_{t_n}\left(x\right)\right)}>\frac{1}{2}\right\}
\end{aligned}$$

and:

$$\begin{aligned}
&\left|\hat{\Phi}_j^{t_n,x}\left(\varphi\right)-\tilde{\Phi}_j^{t_n,x}\left(\varphi\right)\right|\mathbf{1}\left\{\frac{\left|\varepsilon_j^{t_n,x}\left(1\right)\right|}{\mathbb{P}\left(\bar{X}_{t_n}\in B_{t_n}\left(x\right)\right)}\leq\frac{1}{2}\right\} \\
=\ &\left|\hat{\Phi}_j^{t_n,x}\left(\varphi\right)-\tilde{\Phi}_j^{t_n,x}\left(\varphi\right)\frac{\mathbb{P}\left(\bar{X}_{t_n}\in B_{t_n}\left(x\right)\right)}{\frac{1}{M}\sum_{m=1}^{M}\mathbf{1}\left\{\bar{X}_{t_n}^m\in B_{t_n}\left(x\right)\right\}}\right. \\
&\left.-\tilde{\Phi}_j^{t_n,x}\left(\varphi\right)\frac{\varepsilon_j^{t_n,x}\left(1\right)}{\frac{1}{M}\sum_{m=1}^{M}\mathbf{1}\left\{\bar{X}_{t_n}^m\in B_{t_n}\left(x\right)\right\}}\right|\mathbf{1}\left\{\frac{\left|\varepsilon_j^{t_n,x}\left(1\right)\right|}{\mathbb{P}\left(\bar{X}_{t_n}\in B_{t_n}\left(x\right)\right)}\leq\frac{1}{2}\right\} \\
\leq\ &\left\{\frac{\left|\varepsilon_j^{t_n,x}\left(\varphi\right)\right|}{\frac{1}{M}\sum_{m=1}^{M}\mathbf{1}\left\{\bar{X}_{t_n}^m\in B_{t_n}\left(x\right)\right\}}\wedge 3\Gamma^{t_n,x}\left(\varphi\right)\right. \\
&\left.+\left|\tilde{\Phi}_j^{t_n,x}\left(\varphi\right)\right|\frac{\left|\varepsilon_j^{t_n,x}\left(1\right)\right|}{\frac{1}{M}\sum_{m=1}^{M}\mathbf{1}\left\{\bar{X}_{t_n}^m\in B_{t_n}\left(x\right)\right\}}\right\}\mathbf{1}\left\{\frac{\left|\varepsilon_j^{t_n,x}\left(1\right)\right|}{\mathbb{P}\left(\bar{X}_{t_n}\in B_{t_n}\left(x\right)\right)}\leq\frac{1}{2}\right\} \\
\leq\ &\frac{2}{\mathbb{P}\left(\bar{X}_{t_n}\in B_{t_n}\left(x\right)\right)}\left\{\left|\varepsilon_j^{t_n,x}\left(\varphi\right)\right|\wedge 5\Gamma^{t_n,x}\left(\varphi\right)+\left|\varepsilon_j^{t_n,x}\left(1\right)\right|\Gamma^{t_n,x}\left(\varphi\right)\right\}\mathbf{1}\left\{\frac{\left|\varepsilon_j^{t_n,x}\left(1\right)\right|}{\mathbb{P}\left(\bar{X}_{t_n}\in B_{t_n}\left(x\right)\right)}\leq\frac{1}{2}\right\}
\end{aligned}$$

Now, for any $p \geq 1$:

$$\begin{aligned}
&\left|\hat{\Phi}_j^{t_n,x}(\varphi) - \tilde{\Phi}_j^{t_n,x}(\varphi)\right|^p \\
\leq\ & \frac{2^{3p-2}}{\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)^p}\left\{\left\{\left|\varepsilon_j^{t_n,x}(\varphi)\right| \wedge 5\Gamma^{t_n,x}(\varphi)\right\}^p + \left\{\left|\varepsilon_j^{t_n,x}(1)\right| \Gamma^{t_n,x}(\varphi)\right\}^p\right\} \times \\
& \mathbf{1}\left\{\frac{\left|\varepsilon_j^{t_n,x}(1)\right|}{\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)} \leq \frac{1}{2}\right\} + 2^{2p-1}\left(\Gamma^{t_n,x}(\varphi)\right)^p \mathbf{1}\left\{\frac{\left|\varepsilon_j^{t_n,x}(1)\right|}{\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)} > \frac{1}{2}\right\}
\end{aligned}$$

and:

$$\begin{aligned}
&\mathbb{E}\left[\left|\hat{\Phi}_j^{t_n,x}(\varphi) - \tilde{\Phi}_j^{t_n,x}(\varphi)\right|^p\right] \\
\leq\ & \frac{2^{3p-2}}{\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)^p}\left\{\mathbb{E}\left[\left\{\left|\varepsilon_j^{t_n,x}(\varphi)\right| \wedge 5\Gamma^{t_n,x}(\varphi)\right\}^p\right] + \left(\Gamma^{t_n,x}(\varphi)\right)^p \mathbb{E}\left[\left|\varepsilon_j^{t_n,x}(1)\right|^p\right]\right\} \\
& + 2^{2p-1}\left(\Gamma^{t_n,x}(\varphi)\right)^p \mathbb{P}\left(\left|\varepsilon_j^{t_n,x}(1)\right|^p > \frac{\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)^p}{2^p}\right) \\
\leq\ & \frac{8^p}{\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)^p}\left\{\mathbb{E}\left[\left\{\left|\varepsilon_j^{t_n,x}(\varphi)\right| \wedge 5\Gamma^{t_n,x}(\varphi)\right\}^p\right] + \left\{\Gamma^{t_n,x}(\varphi)\right\}^p \mathbb{E}\left[\left|\varepsilon_j^{t_n,x}(1)\right|^p\right]\right\}
\end{aligned}$$

using Markov's inequality. Now, for large M:

$$\begin{aligned}
\mathbb{E}\left[\left|\varepsilon_j^{t_n,x}(1)\right|^p\right] &\leq C_p \frac{\mathrm{Var}\left[\mathbf{1}\left\{\bar{X}_{t_n} \in B_{t_n}(x)\right\}\right]^{\frac{p}{2}}}{M^{\frac{p}{2}}} \\
\mathbb{E}\left[\left|\varepsilon_j^{t_n,x}(\varphi)\right|^p\right] &\leq C_p \frac{\mathrm{Var}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}, i\right)\mathbf{1}\left\{\bar{X}_{t_n} \in B_{t_n}(x)\right\}\right]^{\frac{p}{2}}}{M^{\frac{p}{2}}}
\end{aligned}$$

and:

$$\begin{aligned}
\mathrm{Var}\left[\mathbf{1}\left\{\bar{X}_{t_n} \in B_{t_n}(x)\right\}\right] &= \mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right) - \mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)^2 \\
&\leq \mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}, i\right)\mathbf{1}\left\{\bar{X}_{t_n} \in B_{t_n}(x)\right\}\right] &= \mathbb{E}\left[\varphi^2\left(t_{n+1}, \bar{X}_{t_{n+1}}, i\right)\mathbf{1}\left\{\bar{X}_{t_n} \in B_{t_n}(x)\right\}\right] \\
&\quad -\mathbb{E}\left[\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}, i\right)\mathbf{1}\left\{\bar{X}_{t_n} \in B_{t_n}(x)\right\}\right]^2 \\
&= \tilde{\lambda}_j^{t_n,x}\left(\varphi^2\right)\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right) \\
&\quad -\left\{\tilde{\lambda}_j^{t_n,x}(\varphi)\,\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)\right\}^2 \\
&\leq \tilde{\lambda}_j^{t_n,x}\left(\varphi^2\right)\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)
\end{aligned}$$

Hence, combining these inequalities, $\exists C_p > 0$ such that:

$$\left\|\hat{\Phi}_j^{t_n,x}(\varphi) - \tilde{\Phi}_j^{t_n,x}(\varphi)\right\|_p \leq C_p \frac{\sqrt{\tilde{\lambda}_j^{t_n,x}\left(\varphi^2\right)} + \Gamma^{t_n,x}(\varphi)}{\sqrt{M}\sqrt{\mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}(x)\right)}}$$

□

Unlike equation (3.30), the majoration (3.32) does depend on x, which complicates the assessment of the error between the value functions. This is where the space localization enters into play (cf. Paragraph 3.1.3). Indeed, $\mathcal{D}_{t_n}^{\varepsilon}$ being bounded, one can define:

$$\begin{aligned}
p_{t_n}(\delta, \epsilon) &:= \min_{B_{t_n}^k \subset \mathcal{D}_{t_n}^{\varepsilon}} \mathbb{P}\left(\bar{X}_{t_n} \in B_{t_n}^k\right) > 0 \\
p(\delta, \epsilon) &:= \min_{t_n \in \Pi} p_{t_n}(\delta, \epsilon)
\end{aligned} \tag{3.33}$$

allowing to deal with the denominator. For the numerator, recalling the definition of $C(T, \varepsilon)$ (Paragraph 3.1.3), one can prove the following inequalities:

Lemma 3.7. *For every* $(t_n, x, j) \in \Pi \times \mathcal{D}_\Pi^\varepsilon \times \mathbb{I}_q$*:*

$$\begin{aligned}\Gamma_j^{t_n,x}(\bar{v}_\Pi) &\leq Ce^{-\rho t_n}\left(1 + C(T,\varepsilon) + \sqrt{h}\right)\\ \sqrt{\tilde{\lambda}_j^{t_n,x}(\bar{v}_\Pi^2)} &\leq Ce^{-\rho t_n}\left\{1 + C(T,\varepsilon) + \mathcal{O}(\delta) + \mathcal{O}\left(\sqrt{h}\right)\right\}\end{aligned}$$

In particular:

$$\sqrt{\tilde{\lambda}_j^{t_n,x}(\bar{v}_\Pi^2)} + \Gamma^{t_n,x}(\bar{v}_\Pi) \leq Ce^{-\rho t_n}\left\{1 + C(\varepsilon) + \mathcal{O}(\delta) + \mathcal{O}\left(\sqrt{h}\right)\right\}$$

Proof. For the first inequality, use equation (3.28). For the second, proceed as follows:
Using equation (3.28), $\exists C > 0$ s.t. $\forall (t_n, x, i) \in \Pi \times \mathcal{D}_\Pi^\varepsilon \times \mathbb{I}_q$:

$$\bar{v}_\Pi^2(t_n, x, i) \leq Ce^{-2\rho t_n}\left(1 + |x| + \sqrt{h}\right)^2$$

Then, developing the Euler scheme, one can show that:

$$\mathbb{E}\left[\left(1 + \left|\bar{X}_{t_{n+1}}^{t_n,x}\right| + h\right)^2\right] \leq C(1+|x|)^2\left(1 + \sqrt{h} + h\right)^2$$

Consequently, $\exists C > 0$ s.t. $\forall (t_n, x, j) \in \Pi \times \mathcal{D}_\Pi^\varepsilon \times \mathbb{I}_q$:

$$\left|\Phi_j^{t_n,x}(\bar{v}_\Pi^2)\right| \leq Ce^{-2\rho t_n}(1+|x|)^2\left(1 + \sqrt{h} + h\right)^2 \tag{3.34}$$

Next, we show that $\bar{v}_\Pi^2$ is Lipschitz on $\mathcal{D}^\varepsilon$. Indeed, using Lemma 3.5, Remark 3.5 and equation 3.28, $\forall (t_n, x_1, x_2, i) \in \Pi \times \left(\mathcal{D}_\Pi^\varepsilon\right)^2 \times \mathbb{I}_q$:

$$\begin{aligned}\left|\bar{v}_\Pi^2(t_n,x_1,i) - \bar{v}_\Pi^2(t_n,x_2,i)\right| &= \left|\bar{v}_\Pi(t_n,x_1,i) - \bar{v}_\Pi(t_n,x_2,i)\right|\left|\bar{v}_\Pi(t_n,x_1,i) + \bar{v}_\Pi(t_n,x_2,i)\right|\\ &\leq Ce^{-2\rho t_n}|x_1 - x_2|\left(1 + |x_1| + |x_2| + \mathcal{O}\left(\sqrt{h}\right)\right)\\ &\leq Ce^{-2\rho t_n}|x_1 - x_2|\left(1 + C(T,\varepsilon) + \mathcal{O}\left(\sqrt{h}\right)\right)\end{aligned}$$

Using this result and Lemma 3.4, $\forall (t_n, x, j) \in \Pi \times \mathcal{D}_\Pi^\varepsilon \times \mathbb{I}_q$:

$$\left|\tilde{\lambda}_j^{t_n,x}(\bar{v}_\Pi^2) - \Phi_j^{t_n,x}(\bar{v}_\Pi^2)\right| \leq Ce^{-2\rho t_n}\delta\left(1 + C(T,\varepsilon) + \mathcal{O}\left(\sqrt{h}\right)\right)$$

Combining this result with equation 3.34, $\forall (t_n, x, j) \in \Pi \times \mathcal{D}_\Pi^\varepsilon \times \mathbb{I}_q$:

$$\sqrt{\left|\tilde{\lambda}_j^{t_n,x}(\bar{v}_\Pi^2)\right|} \leq Ce^{-\rho t_n}\left\{1 + C(T,\varepsilon) + \mathcal{O}(\delta) + \mathcal{O}\left(\sqrt{h}\right)\right\}$$

□

Corollary 3.1. *For every* $p \geq 1$*, there exists* $C_p \geq 0$ *s.t.* $\forall (t_n, x, j) \in \Pi \times \mathcal{D}_\Pi^\varepsilon \times \mathbb{I}_q$*:*

$$\left\|\hat{\Phi}_j^{t_n,x}(\bar{v}_\Pi) - \tilde{\Phi}_j^{t_n,x}(\bar{v}_\Pi)\right\|_p \leq C_p e^{-\rho t_n}\frac{1 + C(T,\varepsilon) + \mathcal{O}(\delta) + \mathcal{O}\left(\sqrt{h}\right)}{\sqrt{M}\sqrt{p(\delta,\epsilon)}}$$

Proof. Combine Lemma 3.6, equation (3.33) and Lemma 3.7. □

Proposition 3.7. $\forall p \geq 1$, $\exists C_p > 0$ *s.t.* $\forall (t, x, i) \in \Pi \times \mathcal{D}^\varepsilon \times \mathbb{I}_q$ *:*

$$\left\|\bar{v}_\Pi(t,x,i) - \hat{v}_\Pi(t,x,i)\right\|_p \leq C_p e^{-\rho t_n}\left\{\frac{1 + C(T,\varepsilon) + \mathcal{O}(\delta) + \mathcal{O}\left(\sqrt{h}\right)}{h\sqrt{M}\sqrt{p(\delta,\epsilon)}} + \frac{\delta}{h}\right\}$$

Proof. For each $t_n \in \Pi$, we look for an upper bound E_n, independent of x and i, of the quantity $\left\|\bar{v}_\Pi(t,x,i) - \hat{v}_\Pi(t,x,i)\right\|_p$, $x \in \mathcal{D}^\varepsilon_{t_n}$. First:

$$\left\|\bar{v}_\Pi(T,x,i) - \hat{v}_\Pi(T,x,i)\right\|_p = \left\|g(T,x,i) - g(T,x,i)\right\|_p = 0$$

Hence $E_N = 0$. Fix now $n \in [0, N-1]$. Recall the dynamic programming equations from Remark 3.4, and introduce j^* (resp. $\hat{j}^*$) the arg max for $\bar{v}_\Pi$ (resp. $\hat{v}_\Pi$), i.e.:

$$\begin{aligned}
\bar{v}_\Pi(t_n,x,i) &= hf(t_n,x,j^*) - k(t_n,i,j^*) + \Phi^{t_n,x}_{j^*}(\bar{v}_\Pi) \\
\hat{v}_\Pi(t_n,x,i) &= hf\left(t_n,x,\hat{j}^*\right) - k\left(t_n,i,\hat{j}^*\right) + \hat{\Phi}^{t_n,x}_{\hat{j}^*}(\hat{v}_\Pi)
\end{aligned}$$

Now, for every $\forall (t,x,i) \in \Pi \times \mathcal{D}^\varepsilon_\Pi \times \mathbb{I}_q$:

$$\begin{aligned}
\hat{v}_\Pi(t_n,x,i) &= hf\left(t_n,x,\hat{j}^*\right) - k\left(t_n,i,\hat{j}^*\right) + \hat{\Phi}^{t_n,x}_{\hat{j}^*}(\hat{v}_\Pi) \\
&= hf\left(t_n,x,\hat{j}^*\right) - k\left(t_n,i,\hat{j}^*\right) + \Phi^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) \\
&\quad + \tilde{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) - \Phi^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) \\
&\quad + \hat{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) - \tilde{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) \\
&\quad + \hat{\Phi}^{t_n,x}_{\hat{j}^*}(\hat{v}_\Pi) - \hat{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) \\
&\leq \bar{v}_\Pi(t_n,x,i) + C_{n+1}\delta(1+\theta(h)) \\
&\quad \hat{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) - \tilde{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) \\
&\quad + E_{n+1}
\end{aligned}$$

as D^ε contains all the trajectories ($\bar{X}$ being projected on D^ε). Hence, a.s.:

$$\hat{v}_\Pi(t_n,x,i) - \bar{v}_\Pi(t_n,x,i) \leq \left|\hat{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) - \tilde{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi)\right| + C_{n+1}\delta(1+\theta(h)) + E_{n+1}$$

Symmetrically:

$$\bar{v}_\Pi(t_n,x,i) - \hat{v}_\Pi(t_n,x,i) \leq \left|\tilde{\Phi}^{t_n,x}_{j^*}(\bar{v}_\Pi) - \hat{\Phi}^{t_n,x}_{j^*}(\bar{v}_\Pi)\right| + C_{n+1}\delta(1+\theta(h)) + E_{n+1}$$

Thus:

$$\begin{aligned}
&\left|\bar{v}_\Pi(t_n,x,i) - \hat{v}_\Pi(t_n,x,i)\right| \\
\leq\ & \left|\hat{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) - \tilde{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi)\right| + \left|\hat{\Phi}^{t_n,x}_{j^*}(\bar{v}_\Pi) - \tilde{\Phi}^{t_n,x}_{j^*}(\bar{v}_\Pi)\right| + C_{n+1}\delta(1+\theta(h)) + E_{n+1}
\end{aligned}$$

and, $\forall p \geq 1$:

$$\begin{aligned}
&\left\|\bar{v}_\Pi(t_n,x,i) - \hat{v}_\Pi(t_n,x,i)\right\|_p \\
\leq\ & \left\|\left|\hat{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) - \tilde{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi)\right| + \left|\hat{\Phi}^{t_n,x}_{j^*}(\bar{v}_\Pi) - \tilde{\Phi}^{t_n,x}_{j^*}(\bar{v}_\Pi)\right| + C_{n+1}\delta(1+\theta(h)) + E_{n+1}\right\|_p \\
\leq\ & \left\|\hat{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi) - \tilde{\Phi}^{t_n,x}_{\hat{j}^*}(\bar{v}_\Pi)\right\|_p + \left\|\hat{\Phi}^{t_n,x}_{j^*}(\bar{v}_\Pi) - \tilde{\Phi}^{t_n,x}_{j^*}(\bar{v}_\Pi)\right\|_p + C_{n+1}\delta(1+\theta(h)) + E_{n+1} \\
\leq\ & C_p e^{-\rho t_n} \frac{1 + C(T,\varepsilon) + \mathcal{O}(\delta) + \mathcal{O}\left(\sqrt{h}\right)}{\sqrt{M}\sqrt{p(\delta,\epsilon)}} + C_{n+1}\delta(1+\theta(h)) + E_{n+1}
\end{aligned}$$

for some constant $C_p > 0$ which depends only on p, using Corollary 3.1. By an induction argument:

$$\left\|\bar{v}_\Pi(t_n,x,i) - \hat{v}_\Pi(t_n,x,i)\right\|_p \leq E_n$$

where:

$$\begin{aligned}
E_N &= 0 \\
E_n &= C_p e^{-\rho t_n} \frac{1 + C(T,\varepsilon) + \mathcal{O}(\delta) + \mathcal{O}\left(\sqrt{h}\right)}{\sqrt{M}\sqrt{p(\delta,\epsilon)}} + C_{n+1}\delta(1+\theta(h)) + E_{n+1}
\end{aligned}$$

Consequently:

$$\begin{aligned} E_n &= C_p \frac{1 + C(T,\varepsilon) + \mathcal{O}(\delta) + \mathcal{O}\left(\sqrt{h}\right)}{\sqrt{M}\sqrt{p(\delta,\epsilon)}} \sum_{k=n}^{N} e^{-\rho t_k} + \delta(1+\theta(h)) \sum_{k=n+1}^{N} C_k \\ &\leq_{\sim_{h\to 0}} C_p e^{-\rho t_n} \left\{ \frac{1 + C(T,\varepsilon) + \mathcal{O}(\delta) + \mathcal{O}\left(\sqrt{h}\right)}{h\sqrt{M}\sqrt{p(\delta,\epsilon)}} + \frac{\delta}{h} \right\} \end{aligned}$$

where $C_p > 0$ depends only on p. □

Combining all these results, we can now state our main theorem:

Theorem 3.1. *$\forall p \geq 1$, $\exists C_p > 0$ s.t. $\forall (x,i) \in \mathcal{D}_{t_0}^{\varepsilon} \times \mathbb{I}_q$:*

$$\begin{aligned} &\left\| v(0,x,i) - \hat{v}_{\Pi}(0,x,i) \right\|_p \\ \leq\ & C_p \left\{ (1+|x|) e^{-\bar{\rho}T} + (1+|x|)^{\frac{3}{2}} \sqrt{h} + \varepsilon + \frac{1}{h} \left(\delta + \frac{1 + C(T,\varepsilon) + \mathcal{O}(\delta) + \mathcal{O}\left(\sqrt{h}\right)}{\sqrt{M}\sqrt{p(\delta,\epsilon)}} \right) \right\} \end{aligned}$$

In particular, $\forall (x,i) \in \mathcal{D}_{t_0}^{\varepsilon} \times \mathbb{I}_q$, $\hat{v}_{\Pi}(0,x,i) \to_{L_p} v(0,x,i)$ when $T \to +\infty$, $h \to 0$, $\epsilon \to 0$, $\delta \to 0$ and $M \to +\infty$ with $\frac{\delta}{h} \to 0$ and $\frac{C(T,\varepsilon)}{h\sqrt{M}\sqrt{p(\delta,\epsilon)}} \to 0$.

Proof. Combine Propositions 3.1, 3.2, 3.2, 3.3, 3.6 and 3.7 at $t = 0$. □

Remark 3.6. If the cost function k (recall Assumption 4) were to depend on x, then, under a usual Lipschitz condition on k (similar to that of f), Theorem 3.1 can be shown to be still valid, replacing only the term $(1+|x|)^{\frac{3}{2}} \sqrt{h}$ by $\left(1 + |x|^{\frac{5}{2}}\right) \sqrt{h \log\left(\frac{2T}{h}\right)}$ (recalling Remark 3.2).

4 Complexity analysis and memory reduction

4.1 Complexity

4.1.1 Numerical complexity

The number of operations required by the algorithm described below is in $\mathcal{O}\left(q^2.N.M\log(M)\right)$, where we recall that q is the number of possible switches, N is the number of time steps and M is the number of Monte Carlo trajectories.

The q^2 term stems from the fact that for every $i \in \mathbb{I}_q$, one has to compute a maximum on $j \in \mathbb{I}_q$ (see equation (3.15)). However, this q^2 can be reduced to q as soon as the two following conditions are satisfied:

1. (Irreversibility) The controlled variable can only be increased (or, symmetrically, can only be decreased)

2. (Cost Separability) There exists two functions k_1 and k_2 such that $\forall (t,i,j) \in \mathbb{R}^+ \times \mathbb{I}_q \times \mathbb{I}_q$, $k(t,i,j) = k_1(t,i) + k_2(t,j)$. For instance, this is true of affine costs.

Indeed, under those two conditions, equation (3.15) becomes:

$$\hat{v}_{\Pi}(t_n,x,i) + k_1(t_n,i) = \max_{j \in \mathbb{I}_q,\, j \geq i} \left\{ hf(t_n,x,j) - k_2(t_n,j) + \hat{\mathbb{E}}\left[\hat{v}_{\Pi}\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n,x}, j\right) \right] \right\},\ n = N-1,\ldots,0$$

These maxima can be computed in $\mathcal{O}(q)$ instead of $\mathcal{O}\left(q^2\right)$ by starting from the biggest element $i = i_q$ down to the smallest element $i = i_1$ (in lexicographical order) and keeping track of the partial maxima.

Note that these two conditions hold for the numerical application from Section5, providing the improved complexity $\mathcal{O}(q.N.M\log(M))$.

The N term comes from the backward time induction.

The $M\log(M)$ term is due to the chosen regression basis, ie. the adaptative local polynomial basis. We recall from Bouchard and Warin [2011] how to construct this basis in practice:

1. First, sort the ($d-$dimensional) dataset in each dimension using d quicksorts. This is done in $\mathcal{O}(M\log(M))$.

2. Using this information, cut the dataset in each dimension so as to get hypercubes containing approximately the same number of particles.

3. Then, performing the regression, ie. solving the normal equations, takes a negligible time because of the sparsity of the normal matrix.

We emphasis the importance of the second step, as it defines the probability $p(\varepsilon,\delta)$ from equation (3.33), which should be as large as possible in order to limit the regression error (see Proposition 3.7), ie. the hypercubes should precisely contain approximately the same number of particles. This stresses the soundness of this regression basis.

As noticed in Bouchard and Warin [2011], this clustering is all the more so efficient as the coordinates of the dataset are uncorrelated. However, if some strong correlations between coordinates were to exist, one could, before the regression, perform a rotation into a more suitable basis, reverting to the original basis after the regression.

Compared to the usual regressions with global functions, the advantages are overwhelming, from the improved complexity $\mathcal{O}(M\log(M))$ instead of the usual $\mathcal{O}\big(M^2\big)$, to the numerical stability allowing to increase the number of basis functions without numerical explosion.

4.1.2 Memory complexity

The memory size required for solving optimal switching problems (as well as the simpler American options problems and the more complex BSDE problems) by Monte Carlo methods is often said to be in $\mathcal{O}(N.M)$, because, as the Euler scheme is a forward scheme and the dynamic programming principle is a backward scheme, the storage of the Monte Carlo trajectories seems inescapable. This fact is the major limitation of such methods, as acknowledged in Carmona and Ludkovski [2008] for instance.

As such a complexity would be unbearable in high dimension, we describe below a general memory reduction method to obtain a much more amenable complexity of $\mathcal{O}(N+M)$ (or, more precisely, of $\mathcal{O}(m.N+q.M)$ with $m \ll M$). This improvement really opens the door to the use of Monte Carlo methods for Americal options / optimal switching / BSDEs on high-dimensional practical applications. Note that this tool can be combined with all the existing Monte Carlo backward methods which (seem to) require the storage of all the trajectories.

This tool is limited to Markovian processes however, but, more ofter than not, one can circumvent this restriction by increasing the dimension of the state variable.

4.2 General memory reduction method

4.2.1 Description

The memory reduction method for Monte Carlo pricing of American options was pioneered in Chan et al. [2004] for the geometric Brownian motion, and was subsequently extended to multi-dimensional geometric Brownian motions (Chan et al. [2006]) as well as exponential Lévy processes Chan and Wu [2011]. These papers take advantage of the additive nature of the randomness of the processes considered. However, as briefly hinted in Volpe [2009], the memory reduction trick can be extended to more general processes. In particular, it can be combined with any discretization scheme, for

instance the Euler scheme or Milstein scheme, as long as the value of the stochastic process at one time step can be expressed from its value at the subsequent time step.

From a practical point of view, the production of "random" sequences usually involves wisely chosen deterministic sequences, with statistical properties as close as possible to true randomness (cf. Kroese et al. [2011] for instance for an overview). These sequences can usually be set using a *seed*, i.e. a (possibly multidimensional) fixed value aimed at initializing the algorithm which produces the sequence:

$$\begin{array}{ccccccccc} & & \text{rand}() & & \text{rand}() & & \text{rand}() & & \text{rand}() \\ \{\text{set seed } s\} & \rightarrow & \varepsilon_1 & \rightarrow & \varepsilon_2 & \rightarrow & \cdots & \rightarrow & \varepsilon_n \end{array} \tag{4.1}$$

where the rand() operation consists in advancing to the next element of the sequence. Now two useful aspects can be stressed. The first is that one can usually recover the current seed at any stage of the sequence. The second is that, at any time, if one sets again the seed to, say, the seed s from equation (4.1), then the following elements of the sequence will once again be ε_1, ε_2, ... (equation (4.1)). In other words, one can recover any previously produced subsequence of the sequence $(\varepsilon_n)_{n\geq 1}$, provided one stored beforehand the seed at the beginning of the subsequence. This feature is at the core of the memory reduction method, which we are going to discuss below in a general setting.

Consider a Markovian stochastic process $(X_t)_{t\geq 0}$, for instance the solution of the stochastic differential equation (2.2), recalled below:

$$\begin{aligned} X_0 &= x_0 \in \mathbb{R}^d \\ dX_s &= b(s, X_s)\, ds + \sigma(s, X_s)\, dW_s \end{aligned}$$

The application of the Euler scheme to this equation can be denoted as follows:

$$x^j_{t_{i+1}} = f\left(x^j_{t_i}, \varepsilon^j_i\right) \tag{4.2}$$

$$f(x, \varepsilon) := x + b(t_i, x)\, h + \sigma(t_i, x)\, \varepsilon\sqrt{h} \tag{4.3}$$

where $\forall i \in [0, N-1]$ and $\forall j \in [1, M]$, $\varepsilon^j_i \in \mathbb{R}^d$ is drawn from a d-dimensional Gaussian random variable. Suppose that for any $\varepsilon \in \mathbb{R}^d$, the function $x \mapsto f(x, \varepsilon)$ is invertible (call f_{inv} the inverse). Then, starting from the final value $x^j_{t_N}$ of the sequence (4.2), one can recover the whole trajectory of X:

$$x^j_{t_i} = f_{\text{inv}}\left(x^j_{t_{i+1}}, \varepsilon^j_i\right) \tag{4.4}$$

as long as one can recover the previous draws ε^j_{N-1}, ..., ε^j_0. The following pseudo-code describes an easy way to do it. The first stage corresponds to the Euler scheme, with the addition of the storage of the seeds:

Algorithm 1 Euler Scheme

```
1  % Initialization
2  for j from 1 to M
3          X[j] <- xj
4  end for
5
6  % LOOP 1: Euler scheme
7  for i from 0 to N-1
8     S[i] <- getseed()
9     for j from 1 to M
10       E <- rand(d)
11       X[j] <- f(X[j],E)
12    end for
13 end for
```

At that point, the vector X contains the last values X_T^j, $j = 1, \dots, M$. From this point, one can recover the previous values $X_{t_i}^j$, $i = N-1, \dots, 0$, $j = 1, \dots, M$:

Algorithm 2 Inverse Euler Scheme

```
% LOOP 2: Inverse Euler scheme
for i from N-1 down to 0
   setseed(S[i])
   for j from 1 to M
      E <- rand(d)
      X[j] <- finv(X[j],E)
   end for
end for
```

Inside this last loop, one can perform the estimation of the conditional expectations required by the resolution algorithm of our stochastic control problem (equation (2.10)). Compared to the standard resolution storaging the full trajectories $X_{t_i}^j$, $i = 0, \dots, N$, $j = 1, \dots, M$, the pros and cons are the following:

- The memory needed is brought down from $\mathcal{O}(M \times N)$ to $\mathcal{O}(N)$ (storage of the seeds)
- The number of calls to the rand() function is doubled.

In other words, at the price of doubling the computation time, one can bring down the required memory storage by a factor M, which is a very significant saving. Moreover, the theoretical additional computation time can be insignificant in practice, as the availability of much more physical memory makes the resort to the slower virtual memory much less likely.

Remark 4.1. Even though the storage of the seeds does take $\mathcal{O}(N)$ in memory size, the constant may be much greater than 1. For instance, on Matlab®, a seed from the Combined Multiple Recursive algorithm (refer for instance to Kroese et al. [2011] for a description of several random generators) is made of 12 uint32 (32-bit unsigned integer), a seed from the Multiplicative Lagged Fibonacci algorithm is made of 130 uint64, and a seed from the popular Mersenne Twister algorithm is made of 625 uint32.

In order to relieve the seeds' storage, we provide below a finer memory reduction algorithm, and discuss its potential benefits:

Algorithm 3 General Memory Reduction Method

```
% LOOP 1: Seeds storage
for i from 0 to N–1
   S[i] <– getseed()
   for j from 1 to M
      E <– rand(d)
   end for
end for

% Initialization
for j from 1 to M
        X[j] <– xj
end for

% LOOP 2: Euler scheme
for i from 0 to N–1
   setseed(S[N–i–1])
   for j from 1 to M
      E <– rand(d)
      X[j] <– f(X[j],E)
   end for
end for

setseed(S[0])
free(S)

% LOOP 3: Inverse Euler scheme
for i from N–1 down to 0
   for j from 1 to M
      E <– rand(d)
      X[j] <– finv(X[j],E)
   end for
end for
```

Although this algorithm requires three main loops, it enables to perform the last loop without fiddling the seed of the random generator, and without any vector of seeds locked in memory, which will thus be fully dedicated to the regressions and other resolution operations. Moreover, the first two main loops can be performed beforehand once and for all, storing only the last values of the vector X as well as the first seed $\mathsf{S}[0]$. Finally, if the random generator is able to leapfrop a given number of steps, the first loop can be drastically reduced.

4.2.2 Numerical stability

Theoretically, the trajectories produced by the Euler scheme (4.2) and the inverse Euler scheme (4.4) are exactly the same. In practice however, a discrepancy may appear, the cause of which is discussed below.

On a computer, not all real numbers can be reproduced. Indeed, they must be stored on a finite number of bits, using a predefined format (usually the IEEE Standard for Floating-Point Arithmetic (IEEE 754)). In particular, there exists an incompressible distance $\varepsilon > 0$ between two different numbers stored. This causes rounding errors when performing operations on real numbers.

For instance, consider $x \in \mathbb{R}$ and an invertible function $f : \mathbb{R} \mapsto \mathbb{R}$. Compute $y = f(x)$ and then compute $\hat{x} = f_{\text{inv}}(y)$. One would expect that $\hat{x} = x$, but in practice, because of rounding effects, one may get $\hat{x} = x + \epsilon z$ for a small $\epsilon > 0$, where z is a discrete variable, which can be deemed random,

taking values around zero. This phenomenon is illustrated on Figure 4.1, which displays a histogram of $\hat{x} - x$ for $n = 10^7$ different values of $x \in [0, 1]$ and for the simple linear function $f(x) = 2x + 3$.

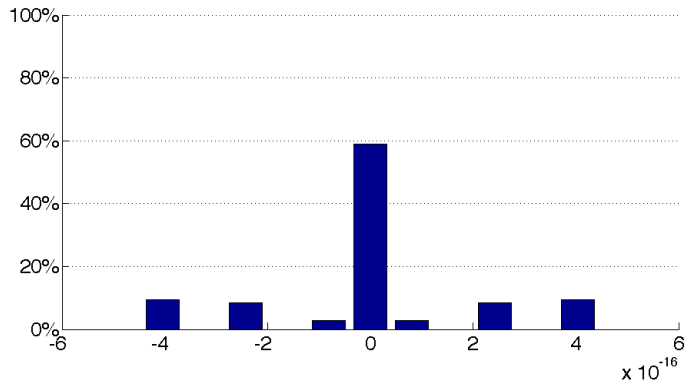


Figure 4.1: Histogram of rounding errors

We now describe how this affects our memory reduction method. Recall equation 4.2:

$$x^j_{t_{i+1}} = f\left(x^j_{t_i}, \varepsilon^j_i\right)$$

Now, instead of equation (4.4), the inverse Euler scheme will provide something like:

$$\begin{aligned} y^j_{t_N} &= x^j_{t_N} \\ y^j_{t_i} &= f_{\text{inv}}\left(y^j_{t_{i+1}}, \varepsilon^j_i\right) + \epsilon z^j_i \end{aligned} \tag{4.5}$$

for a small $\epsilon > 0$, where z^j_i, $i = 0, \ldots, N$, $j = 1, \ldots, M$, can be deemed realizations of a discrete random variable Z, independent of W. The distribution of Z is unknown, but data suggests it may be innocuously assumed centered, symmetric, and with finite moments.

We are now interested in studying the compound rounding error $y_{t_i} - x_{t_i}$ as a function of ϵ. Of course, its behaviour depends on the choice of f (equation (4.3)). Below, we are going to explicit this error on two simple examples to show how greatly its fate can vary.

First example: arithmetic Brownian motion Consider first the case of an arithmetic Brownian motion with drift parameter μ and volatility parameter σ. Here f and its inverse are given by:

$$\begin{aligned} f(x, \varepsilon) &= x + \mu h + \sigma\sqrt{h}\varepsilon \\ f_{\text{inv}}(x, \varepsilon) &= x - \mu h - \sigma\sqrt{h}\varepsilon \end{aligned}$$

Hence, using equation (4.5), for every $j = 1, \ldots, M$:

$$y^j_{t_i} - x^j_{t_i} = \epsilon \sum_{k=i}^{N-1} z^j_k$$

In other words, the compound rounding error behaves as a random walk, multiplied by the small parameter ϵ. Hence, as long as $\epsilon \ll h$ (which is always the case as real numbers smaller than ϵ cannot be handled properly on a computer), this numerical error is harmless.

Remark that a similar numerical error arises from the algorithms proposed in Chan et al. [2004] , Chan et al. [2006] and Chan and Wu [2011], but, fortunately, as discussed above, this error is utterly negligible.

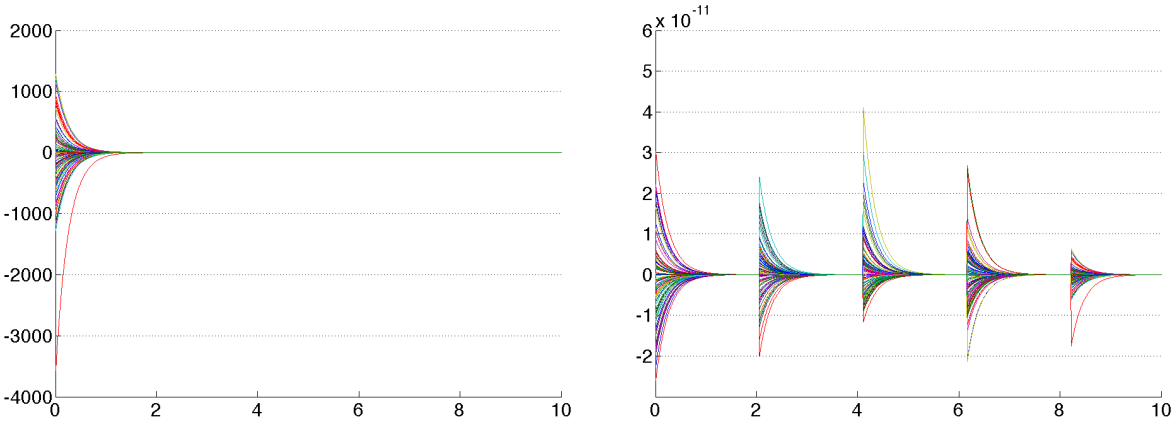
Second example: Ornstein-Uhlenbeck process Now, consider the case of an Ornstein-Uhlenbeck process with mean reversion $\alpha > 0$, long-term mean μ and volatility σ. Here:

$$\begin{aligned} f(x,\varepsilon) &= x + \alpha(\mu - x)h + \sigma\sqrt{h}\varepsilon \\ f_{\text{inv}}(x,\varepsilon) &= \frac{1}{1-\alpha h}\left(x - \alpha\mu h - \sigma\sqrt{h}\varepsilon\right) \end{aligned}$$

Using equation (4.5), for every $j = 1, \dots, M$ the compound error is given by:

$$y_{t_i}^j - x_{t_i}^j = \epsilon \sum_{k=i}^{N-1} \frac{1}{(1-\alpha h)^{k-i}} z_k^j$$

As $(1-\alpha h)^{-N} \sim \exp(\alpha T)$ when $h \to 0$, one can see that, as soon as $T > -\frac{\ln(\epsilon)}{\alpha}$, this error may become overwhelming. This phenomenon is illustrated on Figure 4.2a on a sample of 100 trajectories.



(a) Without intermediate saves

(b) With intermediate saves

Figure 4.2: Compound rounding error for the Ornstein-Uhlenbeck process

In order to mitigate this effect, we propose to modify the Algorithm 3 as follows: in its second loop (usual Euler scheme), instead of saving only the last values x_T^j, one may define a small subset $\tilde{\Pi} \subset \Pi$ and save the intermediate values $x_{t_i}^j$, $t_i \in \tilde{\Pi}$. Then, in the last loop (inverse Euler scheme), every time that $t_i \in \tilde{\Pi}$, the current value of the set $x_{t_i}^j$ may be recovered from this previous storage.

Figure 4.2b illustrates the new behaviour of the compound rounding error with this mended algorithm, on an example with $T = 10$ years and 4 intermediate saves (in addition to the final values).

The drawback of this modification, of course, is that it multiplies the required storage space by the factor $\#\tilde{\Pi}$. However, this remains much smaller than the $\mathcal{O}(M \times N)$ required by the naive full storage algorithm.

5 Application to investment in electricity generation

This section is devoted to an application of the resolution method studied in Section 2. We choose to apply it to an investment problem in electricity generation on a single geographical zone. With this electricity generation investment model, we intend to show how it is possible to provide a probabilistic vision of future electricity generation mixes instead of a deterministic vision provided by plannification methods. Nevertheless, the problem presents so many difficulties that addressing all of them in the same model is unrealistic. Some aspects have thus to be left aside. Since our intention here is to

show that the algorithm described in Section 3 can handle high-dimensional investment problems, our modeling choices were driven by a research for a tractable way to account for the influence of investment decisions on the spot price, in accordance with the fundamentals of the electricity spot price formation mechanism. Although the strategic aspect of investment is an important driver of utilities' decisions, this aspect is beyond the scope of our modeling approach. There exists models in the case of two-stage decision making (see for instance Murphy and Smeers [2005]), but in the case of continuously repeated multiplayer game models, it is even difficult to define closed-loop strategies (see Sec. 2 in Back and Paulsen [2009]). Although also important, we did not consider time-to-build in this implementation. Relying on the fact that it is possible to transform an investment model with time-to-build into a model without time-to-build by replacing capacities with committed capacities (see Bar-Ilan et al. [2002], Aguerrevere [2003] for implementations in dimension one, and Federico and Pham [2012] in dimension two), we left this aspect for future work. Finally, we did not consider the dynamic constraints of power generation. Their effect on spot prices is well-known: they tend to increase spot prices during peak hours and to decrease them during off-peak hours (see Langrené et al. [2011]). However, we assume here that this effect is negligible compared to the effect induced by a lack or an excess of capacity.

Thus, we focused on the following key main drivers of electricity spot prices: demand, capacities (including random outages) and fuel prices. Our model is based on Aïd et al. [2009, 2012], where the electricity spot price is defined as a linear combination of fuel prices multiplied by a scarcity factor. This model exhibits the main feature wanted here, which is that the spot price, being determined both by the fuel prices and the residual capacity, is directly impacted by the evolution of the installed capacity. When the residual capacity tends to decrease, spot prices will tend to increase, making investment worth its cost. Thus, in this model, investments are undertaken not on the specific purpose of satisfying the demand but as soon as they are profitable. In our illustrative example, we start from the situation left by the monopoly period and consider that new capacities are invested according to this process of value maximization. Energy non-served and loss of load probability are outputs that can be adjusted through the price cap on the spot market.

In this section, we first detail the chosen modelling and objective function (which will be shown to be encompassed into the general optimal multiple switching problem (2.1)), and then solve it numerically using the general algorithm developed in the previous sections.

5.1 Modelling

The key variable in order to describe this electricity generation investment problem is the price of electricity. In particular it involves the spread options between the prices of electricity and other energies. In order for the modelling of these options to be accurate, it may be wise to choose a structural model for electricity (cf. the survey Carmona and Coulon [2012]). Hence we choose such a model mainly inspired by those introduced in Aïd et al. [2009] and **?**, albeit amended and customized for a long-term time horizon. All the variables involved are detailed below.

5.1.1 Electricity demand

The electricity demand, or electricity load, at time t on the given geographical zone is denoted as D_t. It is modelled by an exogenous stochastic process $(D_t)_{t\geq 0}$:

$$D_t = f_0(t) + Z_t^0 \tag{5.1}$$

where Z^0 is an Ornstein-Uhlenbeck (henceforth O.U.) process:

$$dZ_t^0 = -\alpha_0 Z_t^0 dt + \beta_0 dW_t^D$$

where α_0 and β_0 are constants, and f_0 is a deterministic function that takes into account demand seasonalities:

$$f_0(t) = d_1 + d_2 \cos\left(2\pi \frac{t - d_3}{l_1}\right) + f_{week}(t) \tag{5.2}$$

where d_j, $1 \leq j \leq 3$ are constants, and, assuming that t is expressed in years, $l_1 = 1$ (yearly seasonality), and f_{week} is a periodic non-parametric deterministic function describing the intra-week load pattern.

5.1.2 Production capacities

Let d' be the number of different production technologies. Denote as $I_t = \left(I_t^1, \ldots, I_t^{d'}\right)$ the installed production capacities. They represent the maximum amount of electricity that is physically possible to produce. These fleets can be increased: at a given time τ_n, one can decide to buy an amount ζ_n of new capacities:

$$I_{\tau_n} = I_{\tau_n^-} + \zeta_n \,,\ n \geq 0 \tag{5.3}$$

Denote as $\alpha = (\tau_n, \zeta_n)_{n\geq 1}$ the corresponding impulse control strategy, where $(\tau_n)_{n\geq 0}$ is an increasing sequence of stopping times, and $(\zeta_n)_{n\geq 0}$ is a sequence of positive vectors corresponding to the increases in capacities. Apart from these increases, I_t will be deemed constant, i.e.:

$$I_t = I_{0-} + \sum_{n,\, \tau_n \leq t} \zeta_n \tag{5.4}$$

Now, denote as $C_t = \left(C_t^1, \ldots, C_t^{d'}\right)$ the available production capacities. Because of spinning reserves, maintenance and random outages, these quantities are lower than the installed capacities I_t, which represent their physical maximum. Said differently, C_t is a fraction of I_t:

$$C_t = I_t \times A_t \tag{5.5}$$

where A_t corresponds to the rate of availability of the different production technologies. Therefore one must choose a model for the process A_t that ensures that it stays within the interval $[0, 1]$.

One possibility would be to model it as a Jacobi process (see for instance Veraart and Veraart [2010] (where it is used to model stochastic correlations) and the references herein for many useful information on this process). This process is however tricky to estimate and simulate (see Gouriéroux and Valéry [2004] for the description of some possible methods), and its main simulation method (the truncated Euler scheme) disables our memory reduction method described in Subsection 4.2. Hence we look for a simpler model.

In Wagner [2012], a detailed structural model for electricity is developed, which included renewable energies like wind and solar. In particular, wind power infeed efficiency (which belongs to $[0, 1]$) is modelled as an Ornstein-Uhlenbeck process with seasonality, mapped into $[0, 1]$ using the logit transform. Adapting this idea, we model $\left(A_t^i\right)_{t\geq 0}^{1\leq i\leq d'}$ as follows:

$$A_t^i := \mathcal{T}\left(f_i\left(t\right) + Z_t^i\right) \tag{5.6}$$

Here Z^i is an O.U. process :

$$dZ_t^i = -\alpha_i Z_t^i dt + \beta_i dW_t^{Z^i}$$

where $\alpha_i > 0$, $\beta_i > 0$ and $\left(W_t^{Z^i}\right)_{t\geq 0}$ is a Brownian motion. The deterministic function f_i accounts for the seasonality in the availability of production capacities:

$$f_i\left(t\right) = c_1^i + c_2^i \cos\left(2\pi \frac{t - c_3^i}{l_1}\right) \tag{5.7}$$

where c_k^i, $1 \leq k \leq 3$, $1 \leq i \leq d'$ are constants. This seasonality stems from the maintenance plannings, which usually mimic the long term seasonality of demand, which in turn originates in the seasonality of temperature. Finally, the function $\mathcal{T} : \mathbb{R} \to [0, 1]$ ensures that $\forall t \geq 0$, $A_t \in [0, 1]^{d'}$. One can choose the versatile logit function as in Wagner [2012], or any other mapping of $\mathbb{R}$ into $[0, 1]$. For instance, any cumulative distribution function would be suitable. As the process Z is gaussian and asymptotically stationary, we choose for $\mathcal{T}$ the (standard) normal cumulative distribution function, as it makes, in particular, the calibration process trivial.

5.1.3 Fuels and CO_2 prices

For each technology, denote as S_t^i the price of the fuel used to produce electricity. In the particular case of renewable energies, which, *per se,* do not involve traded fuels, the corresponding S_t^i can be deemed uniformly null. Moreover, define S_t^0 as the price of CO_2. Denote as S_t the full vector $\left(S_t^0, S_t^1, \ldots, S_t^{d'}\right)$.

Now, we introduce the multiplicative constants needed to convert theses quantities into €/MWh. For each technology $i = 1, \ldots, d'$, let h_i denote its heat rate, and h_i^0 denote its CO_2 emission rate. Hence, the quantity

$$\tilde{S}_t^i := h_i^0 S_t^0 + h_i S_t^i \tag{5.8}$$

expressed in €/MWh, corresponds to the price in € to pay in order to produce 1MWh of electricity using the ith technology. We note $h^0 = \left(h_1^0, \ldots, h_{d'}^0\right) \in \mathbb{R}^{d'}$ and $h = (h_1, \ldots, h_{d'}) \in \mathbb{R}^{d'}$.

Remark 5.1. One can choose to add a fixed cost into the definition of $\tilde{S}_t^i$. This is all the more so relevant for technologies whose fixed costs outweigh the cost of fuel (e.g. nuclear).

Adapting the work of Benmenzer et al. [2007], we model S_t as a multidimensional, cointegrated geometric Brownian motion:

$$dS_t = \Pi S_t dt + \Sigma S_t dW_t^S$$

where Π and Σ are $(d'+1) \times (d'+1)$ matrices with $1 \leq \mathrm{rk}\,(\Pi) < d'$, and $\left(W_t^S\right)_{t \geq 0}$ is a $(d'+1)$-dimensional Brownian motion. This model ensures the positivity of prices, as well as the existence of long-term relationships between energy prices (the relevance of which is illustrated, for instance, in Obermayer [2009]).

5.1.4 Electricity price

As discussed when introducing this section, we build a long-term structural model for the price of electricity. First, we define the marginal cost of electricity using the previously introduced variables. For any time $t \geq 0$, define the permutation $(1), \ldots (M)$ of the numbers $1, \ldots, M$, such that $S_t^{(1)} \leq \ldots \leq S_t^{(M)}$. Then, define $\overline{C}_t^{(i)}$ as the total capacity available at time t from the i first technologies, i.e. $\overline{C}_t^{(i)} := \sum_{j \leq i} C_t^{(j)}$. Using these notations and equation (5.8), the marginal cost of electricity at time t is given by:

$$\begin{aligned} MC_t : &= \widetilde{S}_t^{(1)} \mathbf{1}\left\{D_t < \overline{C}_t^{(1)}\right\} + \sum_{i=2}^{M-1} \widetilde{S}_t^{(i)} \mathbf{1}\left\{\overline{C}_t^{(i-1)} \leq D_t < \overline{C}_t^{(i)}\right\} + \widetilde{S}_t^{(M)} \mathbf{1}\left\{\overline{C}_t^{(M-1)} \leq D_t\right\} \\ &= \widetilde{S}_t^{(1)} + \sum_{i=1}^{M-1} \left(\widetilde{S}_t^{(i+1)} - \widetilde{S}_t^{(i)}\right) \mathbf{1}\left\{D_t - \overline{C}_t^{(i)} \geq 0\right\} \end{aligned}$$

Refer to Aïd et al. [2009] for more details on marginal costs. Remark that the price of CO_2 emissions is explicitly included in the marginal cost (through equation (5.8)).

Now, we are going to use this marginal cost as a building block of our price model, along with some power law *scarcity premiums* (along the lines of **?**) as well as a fixed upper bound [1].

First, consider (x_1, y_1) and (x_2, y_2) two points of $\mathbb{R}^2$. One can always find three positive constants $a := a\,(x_1, x_2, y_1, y_2)$, $b := b\,(x_1, x_2, y_1, y_2)$ and $c := c\,(x_1, x_2, y_1, y_2)$ such that the function:

$$p\,(x) := p\,(x; x_1, x_2, y_1, y_2) = \frac{a}{b - x} + c \tag{5.9}$$

satisfies $p\,(x_1) = y_1$ and $p\,(x_2) = y_2$ (for instance, fix $a > 0$, then define $b = \frac{1}{2}\left(x_1 + x_2 + \sqrt{(x_2 - x_1)^2 + 4a\frac{x_2 - x_1}{y_2 - y_1}}\right)$ and finally $c = y_1 - \frac{a}{b - x_1}$).

[1] Indeed, in the French, German and Austrian markets for instance, power prices cannot be set outside the $[-3000, 3000]$€/MWh range, see `http://www.epexspot.com/en/product-info/auction`.

Using this notation, introduce the price P_t of electricity, defined as follows:

$$\begin{aligned} P_t \quad := \quad & \widetilde{S}_t^{(1)} \mathbf{1}\left\{D_t < 0\right\} + \left\{\widetilde{S}_t^{(1)} + p\left(D_t; 0, \overline{C}_t^{(1)}, \widetilde{S}_t^{(1)}, \widetilde{S}_t^{(2)}\right)\right\} \mathbf{1}\left\{0 \leq D_t < \overline{C}_t^{(1)}\right\} \\ & \sum_{i=2}^{d'-1} \left\{\widetilde{S}_t^{(i)} + p\left(D_t; \overline{C}_t^{(i-1)}, \overline{C}_t^{(i)}, \widetilde{S}_t^{(i)}, \widetilde{S}_t^{(i+1)}\right)\right\} \mathbf{1}\left\{\overline{C}_t^{(i-1)} \leq D_t < \overline{C}_t^{(i)}\right\} \\ & + \left\{\widetilde{S}_t^{(d')} + p\left(D_t; \overline{C}_t^{(d'-1)}, \overline{C}_t^{(d')}, \widetilde{S}_t^{(d')}, M_{\max}\right)\right\} \mathbf{1}\left\{\overline{C}_t^{(d'-1)} \leq D_t\right\} \end{aligned} \tag{5.10}$$

where $M_{\max} > 0$ is a fixed upper bound on the price of electricity. In particular, the last term, the one involving $M_{\max}$, enables price spikes to occur (when the residual capacity is small). Moreover, thanks to the knitting function (5.9), the electricity price P is a Lipschitz continuous function of the structural variables D, C and S. (Rigorously, this property requires that C does not reach zero. One can, for instance, add a fixed minimum availability rate $1 \gg a_{\min} > 0$ to the definition (5.6), replacing $\mathcal{T}$ by $a_{\min} + (1 - a_{\min})\,\mathcal{T}$).

5.1.5 Objective function

We now explicit the objective function of the investor in electricity generation. Suppose that, at time t, an investor builds the quantity ζ of new power generation capacity from the technology $i \in [1, d']$. It generates the cost:

$$\kappa_i^f + \zeta \kappa_i^p$$

where κ_i^f and κ^p are the fixed and proportional costs of building new plants for the technology i. Furthermore, it increases the installed capacity of the type i from I_{t-} to $I_s = I_{t-} + \zeta$, $s \geq t$ (the possible closure of old power plants is not taken into account here). Assuming that the global availability rate(5.6) of technology i applies to the new plants, they can then produce at most ζA_s^i, $s \geq t$, or, more precisely, according to the stack order principle:

$$\min\left\{\zeta A_s^i,\, D_s - \overline{C}_s^{(i-1)}\right\}$$

assuming that, in the stack order, the new plants are called before the older plants I_{t-} of the same technology (as they can be expected to have an at least slightly better efficiency rate compared to the older plants of the same technology, a phenomenon that the function (5.9) aims at capturing). At time $s \geq t$, this production is sold at the price P_s, but costs $\tilde{S}_s$ to produce (if $P_s < \tilde{S}_s$, then of course the producer chooses not to produce). In addition, regardless of the actual production, there may exist a fixed maintenance cost κ_i. Summing up all these gains, discounted to time t using at fixed interest rate $\rho > 0$, the new plants yield a revenue of:

$$\int_t^\infty e^{-\rho s} \min\left\{\zeta A_s^i,\, D_s - \overline{C}_s^{(i-1)}\right\} \times \left(\left(P_s - \widetilde{S}_s\right)^+ - \kappa_i\right) ds$$

This was the cost-benefit analysis for one quantity ζ of new plants. Now, consider as a whole the full fleet of the geographical zone considered. Maximizing the expectation of gains along the potential new plants yields the following value function:

$$v\left(t, x, i\right) = \sup_{\alpha \in \mathcal{A}_{t,i}} \mathbb{E}\left[\sum_{i=1}^{d'} \int_t^\infty e^{-\rho s} \min\left\{C_s^i, D_s - \overline{C}_s^{(i-1)}\right\} \times \left(\left(P_s - \widetilde{S}_s^i\right)^+ - \kappa_i\right) ds - \sum_{\tau_n \geq t} e^{-\rho \tau_n} \left(\kappa_i^f + \zeta_n^i . \kappa_i^p\right)\right] \tag{5.11}$$

using implicitly equation (5.4) (in particular the new plants do alter the law of P), and where purposely discouted all the cash flows up to time 0, the time of interest.

Remark 5.2. Replacing P in (5.11) by its definition (5.10), it is patent that this objective function fits into the mould studied thoroughly in Section 2. In Subsection 5.2 below, the algorithm developed there will be applied to this specific objective function.

Remark 5.3. From an economical point of view, the objective function (5.11) may seem awkward, as on the one hand plants are valued against market prices (as in deregulated markets), but on the other hand, all the plants are considered as a whole (as in a monopolistic market), wrongly neglecting the effect of competition. Indeed, the focus here is brought on the feasibility of the application of the probabilistic algorithm from Section 2 to a non-trivial multidimensional problem. The adaptation of this work to the study of a multiple-agents stochastic control problem is for now left for further research.

5.2 Numerical results

Finally, we solve the control problem described in Subsection 5.1 on a numerical example, using the algorithm detailed in Subsection 3 combined with the general memory reduction method described in Subsection 4.2. Our numerical example encompasses two cointegrated fuels (in addition to the price of CO_2): one "base fuel" and one "peak fuel", starting respectively from 40€/MWh and 80€/MWh. Hence, using the notations from Subsection 5.1, $d' = 2$ (two technologies) and $d = 6$ (electricity demand, CO_2 price, two fuel prices and two availability rates). In order to take into account the minimum size of one power plant we restrict the values of the installed capacity process(5.4) into a (bi-dimensional) fixed grid $\Lambda^{d'}$, with a mesh of 1GW.

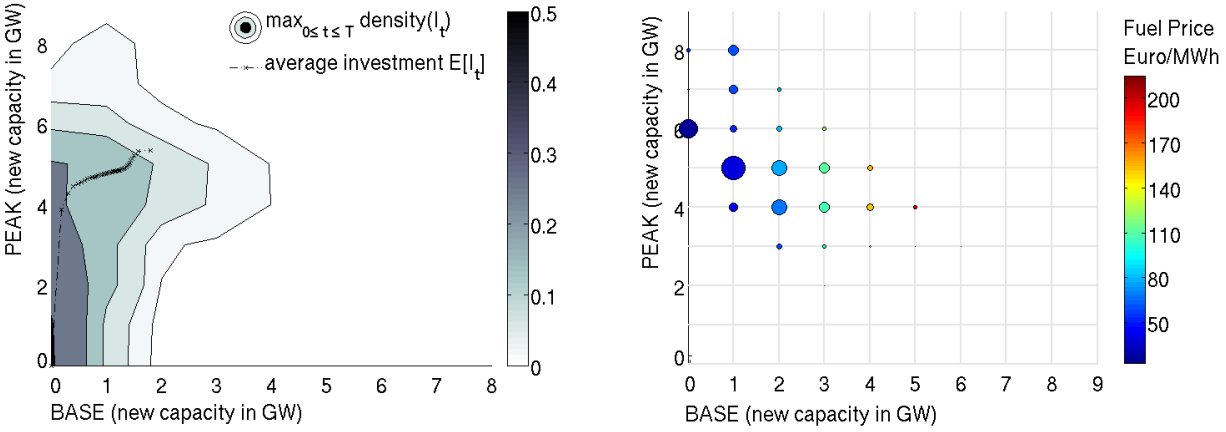
Remark 5.4. If such a grid is indeed manageable in dimension $d' = 2$, it may less be the case if additional technologies were considered. However, as discussed in Tan [2011] (equation (3.2)), instead of performing one regression for each $i \in \Lambda^{d'}$, one can solve equation (3.15) at time t_i by only one $(d + d')$-dimensional regression, by choosing an a priori law for the randomized control ζ_{t_i}. The error analysis from Section 2 can be easily generalized to such regressions in higher dimension.

Figure 5.1 illustrates the evolution of the optimal strategy on this numerical example.

Figure 5.1a shows the time evolution of the average optimal strategy. One can distinguish a first short phase characterised by the construction of several GW of peak load assets, followed by a much slower second phase involving the construction of both base load and peak load assets. Moreover, the increase in variance of the optimal strategy is also clearly visible.

Figure 5.1b details the distribution of the optimal strategy at time $T = 40$ years. One can see that the more the peak fuel is expensive (and hence both fuels are expensive, as they are cointegrated), the more constructions of base load plants occur.

The fact that the average fleet seem to converge is related to the fact that this numerical example does not consider any growth trend in the electricity demand (recall equation (5.2)). Otherwise, more investments would occur, indeed, to keep the pace with consumption.



(a) Time evolution of new capacity strategies

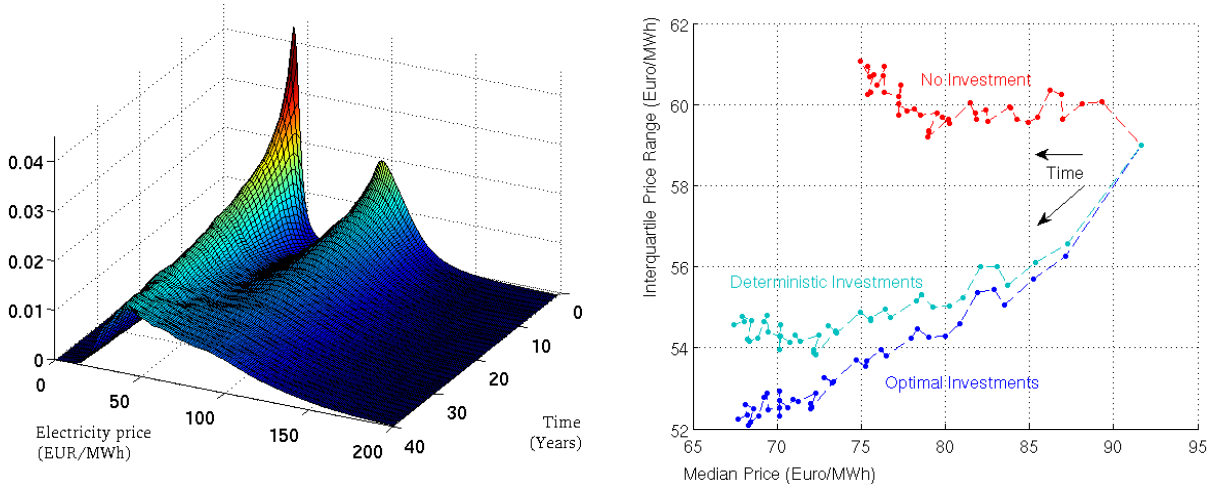
(b) Final fleet distribution

Figure 5.1: Optimal strategies

Figure 5.2 illustrates the evolution of the price of electricity.

Figure 5.2a displays the time evolution of the electricity spot price density. For better readability, each density covers one whole year. One can see how the density moves away from the initial bimodal density (with prices clustering around the initial prices of the two fuels) towards a more diffuse density. One can notice the downward effect of investments on prices.

This effect is more visible on Figure 5.2b. It compares the effect on electricity prices of three different strategies: the optimal strategy, the optimal deterministic strategy (computed as the average of the optimal strategy), and the do-nothing strategy. For each strategy, the joint time-evolution of the yearly median price and the yearly interquartile range are drawn. As expected, prices tend to be higher and more scattered without any new plant. Nevertheless, on this example, the distributions of prices under the optimal deterministic and optimal strategies are close (only slightly less scattered under the optimal strategy).



(a) Time evolution of electricity spot price density

(b) Comparison between investment strategies

Figure 5.2: Electricity spot price

These few pictures give examples on the kind on information that can be be extracted from the resolution of this control problem. Of course, much more can be extracted and analyzed (distribution of income, CO_2 emissions, optimal exercise frontiers, etc).

6 Conclusion

In this paper, we presented a probabilistic method to solve optimal multiple switching problems. We showed on a realistic investment model for electricity generation that it can efficiently provide insight into the distribution of future generation mixes and electricity spot prices. We intend to develop this work in several directions in the future. First, we wish to take into account more generation technologies, most notably wind farms, nuclear production, as well as solar distributed production. These additions would raise the dimension of the problem from eight to fifteen. Yet another range of innovations in numerical methods will be necessary to overcome this increase in dimension. Second, we wish to take time-to-build into account. And last but not least, we wish to adapt the problem to a continuous-time multiplayer game and contribute to the quest for an efficient algorithm to solve it.

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