Calibration of a multifactor model for the forward markets of several commodities

Enrico Edoli*[1,2], Davide Tasinato[1], and Tiziano Vargiolu†[1]

[1]Department of Mathematics, via Trieste 63, University of Padova, I-35121 Padova, Italy
[2]Finalyst SAS, via Trentino 28, I-30010 Campolongo Maggiore, Italy

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Abstract

We propose a model for the evolution of forward prices of several commodities, which is an extension of the two-factor forward model by Kiesel *et al.* (2009), originally conceived for the electricity forward market, to a market where multiple commodities are traded. We then show how to calibrate this model in a market where few or no derivative assets on forward contracts are present. We thus perform a calibration based on historical forward prices. First we calibrate separately the four coefficients of every single commodities, using an approach based on quadratic variation. Then we pass to estimate the mutual correlation among the Brownian motions driving the different commodities, the estimates being based now on the quadratic covariation between forward prices of different commodities. This calibration is compared to a calibration method used by practitioners, which uses rolling time series, which however requires a modification of the model.

Keywords: two-factor model for forward prices, historical calibration, quadratic variation, quadratic covariation.

1 Introduction

When dealing with forward prices of a single commodity having different maturities, the two-factor model proposed by Kiesel *et al.* [4] is quite simple to understand, analytically tractable and gives a good fit of several stylized fact. The first is the so-called Samuelson effect, i.e. the local volatility of a short-term forward contract is greater than the local volatility of a long-term contract, and in particular an exponential decay is observed as the time to maturity of the contract grows. The second stylized fact

*enricoedoli@gmail.com
†Corresponding Author, +39 049 8271383 vargiolu@math.unipd.it.

is that this volatility does not go to zero, but rather to a fixed value, called long-term volatility, due to long term uncertainty factors like technological innovations, change in geo-political equilibria, structural modifications to commodity prices, and so on. Moreover, the model is consistent with market data and with the Schwarz-Smith model for the spot price [6], which exibits mean reversion, another stylized fact which is observed in the markets.

We extend this model by assuming to have $K \geqslant 2$ commodities in our market, and that, for each one of the commodity, their forward prices follow the following two-dimensional model: by denoting with $F^k(t,T)$ the price at time t of a forward contract on the commodity $k = 1, \dots, K$ with maturity T, we assume that under a forward-neutral probability measure $\mathbb{Q}_T$ its dynamics are

$$dF^k(t,T) = F^k(t,T)(e^{-\lambda^k(T-t)}\sigma_1^k \, dW_1^k(t) + \sigma_2^k \, dW_2^k(t))$$

where W_1^k and W_2^k are two correlated Brownian motions with correlation ρ^k and the other parameters represent, respectively:

- σ_1^k - spot volatility, i.e. how much the forward price is influenced by short period shocks;
- σ_2^k - long term volatility, i.e. how much the forward price is influenced by long period uncertainty;
- λ^k - mean-reversion speed, or speed of decaying of the spot volatility.

Thus, when fitting this model to the market data of the k-th commodity, we have to calibrate the four parameters $p^k = (\sigma_1^k, \sigma_2^k, \lambda^k, \rho^k)$. Moreover, we assume that the Brownian motions of the commodities also have an inter-commodity correlation, given by the correlation matrix

$$\rho_{a,b}^{k,m} := corr(W_a^k(t), W_b^m(t)) = \text{Cov}\,(W_a^k(t), W_b^m(t))/t, \qquad \text{i.e.} \qquad \rho_{a,b}^{k,m} := \text{Cov}\,(W_a^k(1), W_b^m(1))$$

for all $a, b = 1, 2$, $k, m = 1, \dots, K$: of course

$$\rho_{1,2}^{k,k} = \rho_{2,1}^{k,k} = \rho^k$$

Thus, the $2K$-dimensional Brownian motion $(W_1^1, W_2^1, \dots, W_1^K, W_2^K)$ has correlation matrix

$$\boldsymbol{\rho} = (\boldsymbol{\rho}^{k,m})_{1 \leqslant k,m \leqslant K} := \begin{pmatrix} \boldsymbol{\rho}^{1,1} & \cdots & \boldsymbol{\rho}^{1,K} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\rho}^{K,1} & \cdots & \boldsymbol{\rho}^{K,K} \end{pmatrix}$$

where

$$\boldsymbol{\rho}^{k,m} = (\rho_{a,b}^{k,m})_{1 \leqslant a,b \leqslant 2} := \begin{pmatrix} \rho_{1,1}^{k,m} & \rho_{1,2}^{k,m} \\ \rho_{2,1}^{k,m} & \rho_{2,2}^{k,m} \end{pmatrix}$$

Recall that, being $\boldsymbol{\rho}$ a correlation matrix, it is symmetric, semi-positive definite, with $\rho_{a,a}^{k,k} = 1$ for all $k = 1, \ldots, K$ and $a = 1, 2$ and $\rho_{1,1}^{k,m} \in [-1, 1]$ for all $k, m = 1, \ldots, K$ and $a, b = 1, 2$.

This model is analytically tractable because, under the forward measure $\mathbb{Q}_T$, each $F^k(\cdot, T)$ has a lognormal evolution, given by

$$F^k(t,T) = F^k(t_0,T) \exp\left(\int_{t_0}^{t} e^{-\lambda^k (T-s)} \sigma_1^k \, dW_1^k(s) + \int_{t_0}^{t} \sigma_2^k \, dW_2^k(s) - \frac{1}{2} \int_{t_0}^{t} \Sigma^k(s,T)^2 \, ds \right)$$

where $\Sigma_k(s, T)$ is a sort of local volatility at time s, given by

$$\Sigma^k(s,T) := \sqrt{e^{-2\lambda^k (T-s)} (\sigma_1^k)^2 + 2\rho^k e^{-\lambda^k (T-s)} \sigma_1^k \sigma_2^k + (\sigma_2^k)^2}$$

Thus, $\log F^k(t, T)$ has a Gaussian distribution, with mean

$$\mathbb{E}_{\mathbb{Q}_T}[\log F^k(t,T)] = \log F^k(t_0,T) - \frac{1}{2} \int_{t_0}^{t} \Sigma_k^2(s,T) \, ds$$

and variance

$$\text{Var}_{\mathbb{Q}_T}[\log F^k(t,T)] = \int_{t_0}^{t} \Sigma_k^2(s,T) \, ds$$

In this paper we want to calibrate this model in a situation where, for each commodity $k = 1, \ldots, K$, forward contracts with (a finite number of) different maturities $T_1^k, \ldots, T_{N_k}^k$ are present, and few or no derivatives on these forward contracts are traded, as can be the case of some markets and/or some commodities. We thus perform a calibration based on historical forward prices. The strategy is first to calibrate separately the four coefficients of every single commodities, as we want them to have priority and greater precision than the correlations among different commodities: in fact, the main aim of our calibration is that it should reproduce well first of all the price behaviour of single-commody products. Secondly, we estimate the correlation matrix also in the inter-commodity correlations.

More in detail, Section 2 shows the calibration procedure of the four parameters of a single commodity, with an approach based on quadratic variation-covariation. Section 3 shows the calibration procedure of the residual parameters, i.e. the inter-commodity correlations, again with an approach based on quadratic covariation. Section 4 present an alternative calibration method which is mostly used by practitioners and uses rolling time series: this method is simpler but, to be made rigorous, it requires to work with a modified model. Section 5 shows empirical findings on the European market.

2 Single commodity calibration

We now fix the commodity $k = 1, \ldots, K$ and assume that, as already mentioned in the Introduction, we have a market where forward contracts with maturities $T_1, \ldots, T_N$

are traded (in this section we omit the dependences on k of the maturities). Then, by denoting $X_i^k(t) := \log F^k(t, T_i)$, we have that

$$dX_i^k(t) = e^{-\lambda^k(T-t)}\sigma_1^k\, dW_1^k(t) + \sigma_2^k\, dW_2^k(t) + \text{ drift}$$

under the forward-neutral probability $\mathbb{Q}_T$. Since we want to perform an historical calibration, we need dynamics under the real world probability $\mathbb{P}$. By the Girsanov theorem, the dynamics of X_i^k under $\mathbb{P}$ is given by

$$dX_i^k(t) = e^{-\lambda^k(T-t)}\sigma_1^k\, d\tilde{W}_1^k(t) + \sigma_2^k\, d\tilde{W}_2^k(t) + \text{ drift}$$

where $\tilde{W}_1^k$ and $\tilde{W}_2^k$ are Brownian motions under $\mathbb{P}$, still with mutual correlation ρ^k, but the drift in the two dynamics are different, as in the second drift also the market price of risk is present. We notice that the coefficients $p^k = (\sigma_1^k, \sigma_2^k, \lambda^k, \rho^k)$ can be estimated directly under $\mathbb{P}$. A more direct writing of the dynamics of X_i^k under $\mathbb{P}$ is

$$dX_i^k(t) = \Sigma_i^k(t)\, d\overline{W}^k(t) + \text{ drift}$$

where

$$\Sigma_i^k(t) := \Sigma^k(s, T_i) = \sqrt{e^{-2\lambda^k(T-t)}(\sigma_1^k)^2 + 2\rho^k e^{-\lambda^k(T-t)}\sigma_1^k\sigma_2^k + (\sigma_2^k)^2}$$

and $\overline{W}^k$ is a suitable 1-dimensional Brownian motion under $\mathbb{P}$.

The fact that the diffusion coefficient of the X_i^k, $i = 1, \ldots, N$, under $\mathbb{P}$ is deterministic gives us a easy way to estimate the parameters. In fact, the quadratic variation of X_i^k under $\mathbb{P}$ is given by

$$\langle X_i^k \rangle_{t_0}^t := \lim_{n\to\infty} \sum_{l=1}^{n} (X_i^k(t_{l+1}) - X_i^k(t_l))^2 = \int_{t_0}^{t} (\Sigma_i^k(u))^2\, du \tag{1}$$

where $t_0 < t_1 < \ldots < t_n = t$ are suitable sequences, and the quadratic covariation of X_i^k, X_j^k, always under $\mathbb{P}$, is given by

$$\langle X_i^k, X_j^k \rangle_{t_0}^t := \lim_{n\to\infty} \sum_{l=1}^{n} (X_i^k(t_{l+1}) - X_i^k(t_l))(X_j^k(t_{l+1}) - X_j^k(t_l)) = \int_{t_0}^{t} \Sigma_{i,j}^k(u)\, du \tag{2}$$

(for more details, see [5]). Now, the last term of these equalities is explicitly computable ($\Sigma_{i,j}^k(u)$ will be specified later in the next Lemma 2.2), while the middle term can be approximated with historical observations. This gives us an idea to calibrate the model: given the historical quadratic covariations, our aim is to find coefficients p^k such that the theoretical quadratic covariations of all forward contracts match as close as possible the historical quadratic covariations.

In order to do this, we must calculate analytically the integrals in Equations (1–2).

Lemma 2.1 *The quadratic variation of the process X_i^k is given by*

$$\begin{aligned}
\langle X_i^k \rangle_{T_0^i}^{T_i^1} &= \int_{T_0^i}^{T_i^1} (\Sigma_i^k(u))^2 \, du = \\
&= \frac{\left(\sigma_1^k\right)^2}{2\lambda^k} \left(e^{-2\lambda^k\left(T_i - T_i^1\right)} - e^{-2\lambda^k\left(T_i - T_i^0\right)}\right) + \left(\sigma_2^k\right)^2 \left(T_i^1 - T_i^0\right) + \\
&\quad + \frac{2\sigma_1^k \sigma_2^k \rho_{1,2}^{k,k}}{\lambda^k} \left(e^{-\lambda^k\left(T_i - T_i^1\right)} - e^{-\lambda^k\left(T_i - T_i^0\right)}\right)
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\int_{T_i^0}^{T_i^1} \left(\Sigma_i^k(t)\right)^2 dt &= \int_{T_i^0}^{T_i^1} \left(e^{-2\lambda^k (T_i - t)} \left(\sigma_1^k\right)^2 + \left(\sigma_2^k\right)^2 + 2e^{-\lambda^k (T_i - t)} \sigma_1^k \sigma_2^k \rho_{1,2}^{k,k}\right) dt \\
&= \left(\sigma_1^k\right)^2 \left[e^{-2\lambda^k (T_i - t)}\right]_{T_i^0}^{T_i^1} + \left(\sigma_2^k\right)^2 \left(T_i^1 - T_i^0\right) + 2\sigma_1^k \sigma_2^k \rho_{1,2}^{k,k} \left[e^{-\lambda^k (T_i - t)}\right]_{T_i^0}^{T_i^1} \\
&= \frac{\left(\sigma_1^k\right)^2}{2\lambda^k} \left(e^{-2\lambda^k\left(T_i - T_i^1\right)} - e^{-2\lambda^k\left(T_i - T_i^0\right)}\right) + \left(\sigma_2^k\right)^2 \left(T_i^1 - T_i^0\right) + \\
&\quad + \frac{2\sigma_1^k \sigma_2^k \rho_{1,2}^{k,k}}{\lambda^k} \left(e^{-\lambda^k\left(T_i - T_i^1\right)} - e^{-\lambda^k\left(T_i - T_i^0\right)}\right)
\end{aligned}$$

□

Lemma 2.2 *The quadratic covariation of the processes X_i^k, X_j^k is given by*

$$\begin{aligned}
\langle X_i^k, X_j^k \rangle_{T_0^i}^{T_i^1} &= \left(\sigma_2^k\right)^2 \left(T_{i,j}^1 - T_{i,j}^0\right) - \frac{e^{-\lambda^k (T_i + T_j)} \left(\sigma_1^k\right)^2}{2\lambda^k} \left(e^{2\lambda^k T_{i,j}^1} - e^{2\lambda^k T_{i,j}^0}\right) + \\
&\quad + \frac{\sigma_1^k \sigma_2^k \rho_{1,2}^{k,k} \left(e^{-\lambda^k T_i} + e^{-\lambda^k T_j}\right)}{\lambda^k} \left(e^{\lambda^k T_{i,j}^1} - e^{\lambda^k T_{i,j}^0}\right)
\end{aligned}$$

Proof. The best way to proceed is to use the so-called polarization identity

$$2 \left\langle X_i^k, X_j^m \right\rangle_{T_{i,j}^0}^{T_{i,j}^1} = \left(\left\langle X_i^k + X_j^m \right\rangle_{T_{i,j}^0}^{T_{i,j}^1} - \left\langle X_i^k \right\rangle_{T_{i,j}^0}^{T_{i,j}^1} - \langle X_j^m \rangle_{T_{i,j}^0}^{T_{i,j}^1} \right) \tag{3}$$

(which will valid also for inter-commodity covariations), where the only missing thing here is $\left\langle X_i^k + X_j^k \right\rangle_{T_{i,j}^0}^{T_{i,j}^1}$: in order to calculate this, first we obtain the stochastic differential of $X_i^k + X_j^k$ as

$$d\left(X_i^k + X_j^k\right) = \left(e^{-\lambda^k T_i} + e^{-\lambda^k T_j}\right) e^{\lambda^k t} \sigma_1^k dW_1^k(t) + 2\sigma_2^k dW_2^k(t) + \text{drift}$$

The variation $\left\langle X_i^k + X_j^k \right\rangle_{T_{i,j}^0}^{T_{i,j}^1}$ is then equal to

$$\begin{aligned}\left\langle X_i^k + X_j^k \right\rangle_{T_{i,j}^0}^{T_{i,j}^1} &= \int_{T_{i,j}^0}^{T_{i,j}^1} \left(e^{-\lambda^k T_i} + e^{-\lambda^k T_j}\right)^2 e^{2\lambda^k t} \left(\sigma_1^k\right)^2 + 4\left(\sigma_2^k\right)^2 + 4\sigma_1^k \sigma_2^k \rho_{1,2}^{k,k} \left(e^{-\lambda^k T_i} + e^{-\lambda^k T_j}\right) e^{\lambda^k t}\, dt \\ &= \frac{\left(e^{-\lambda^k T_i} + e^{-\lambda^k T_j}\right)^2 \left(\sigma_1^k\right)^2}{2\lambda^k} \left(e^{2\lambda^k T_{i,j}^1} - e^{2\lambda^k T_{i,j}^0}\right) + 4\left(\sigma_2^k\right)^2 \left(T_{i,j}^1 - T_{i,j}^0\right) + \\ &\quad + \frac{4\sigma_1^k \sigma_2^k \rho_{1,2}^{k,k} \left(e^{-\lambda^k T_i} + e^{-\lambda^k T_j}\right)}{\lambda^k} \left(e^{\lambda^k T_{i,j}^1} - e^{\lambda^k T_{i,j}^0}\right)\end{aligned}$$

By putting all together, the result is

$$\begin{aligned}2\left\langle X_i^k, X_j^k \right\rangle_{T_{i,j}^0}^{T_{i,j}^1} &= 2\left(\sigma_2^k\right)^2 \left(T_{i,j}^1 - T_{i,j}^0\right) - \frac{e^{-\lambda^k (T_i + T_j)} \left(\sigma_1^k\right)^2}{\lambda^k} \left(e^{2\lambda^k T_{i,j}^1} - e^{2\lambda^k T_{i,j}^0}\right) + \\ &\quad + \frac{2\sigma_1^k \sigma_2^k \rho_{1,2}^{k,k} \left(e^{-\lambda^k T_i} + e^{-\lambda^k T_j}\right)}{\lambda^k} \left(e^{\lambda^k T_{i,j}^1} - e^{\lambda^k T_{i,j}^0}\right)\end{aligned}$$

which gives the desired result □

As already pointed out, our strategy is to have the model quadratic covariations $\left\langle X_i^k, X_j^k \right\rangle_{T_{i,j}^0}^{T_{i,j}^1}$ as close as possible to the market quadratic covariations, which are estimated using the **realized variation estimators**

$$\overline{\left\langle X_i^k \right\rangle_{T_i^0}^{T_i^1}} := \sum_{j=1}^{n} \left(X_i^k (t_{j+1}) - X_i^k (t_j)\right)^2$$

and the **realized covariation estimators**

$$\overline{\left\langle X_i^k, X_j^m \right\rangle_{T_{i,j}^0}^{T_{i,j}^1}} := \sum_{l=1}^{n} \left(X_i^k (t_{l+1}) - X_i^k (t_l)\right) \left(X_j^m (t_{l+1}) - X_j^m (t_l)\right)$$

(which in this section we will use only with $k = m$). Ideally, we would impose that

$$\overline{\left\langle X_i^k, X_j^k \right\rangle_{T_{i,j}^0}^{T_{i,j}^1}} = \left\langle X_i^k, X_j^k \right\rangle_{T_{i,j}^0}^{T_{i,j}^1} \qquad \text{for all } i, j = 1, \ldots, N_k$$

However, the second terms of this system depend only on the four parameters $p^k = (\sigma_1^k, \sigma_2^k, \lambda^k, \rho^k)$, so the system is likely to be overdetermined for $N_k > 2$. For this reason, we estimate the four parameters with a mean-square estimation, i.e. define $\hat{p}^k$ as the 4-dimensional vector which solves

$$\min_{p^k} \sum_{i,j=1}^{N_k} \left(\overline{\left\langle X_i^k, X_j^k \right\rangle_{T_{i,j}^0}^{T_{i,j}^1}} - \left\langle X_i^k, X_j^k \right\rangle_{T_{i,j}^0}^{T_{i,j}^1}\right)^2$$

In this way we obtain all the parameters p^k for all the single commodities, while the inter-commodity correlations $(\rho_{a,b}^{k,m})_{a,b=1,2,k\neq m}$ still remain to be estimated.

3 Calibration of the intercommodity correlations

In order to calibrate for the intercommodity correlations, we continue to use the idea of using the quadratic covariations among the log-forward prices X_k^i for $k = 1, \ldots, K$ and $i = 1, \ldots, N_k$. In fact, for all suitable i, j, k, m, the quadratic covariations of X_k^i, X_m^j is given by

$$\langle X_i^k, X_j^m \rangle_{t_0}^t := \lim_{n\to\infty} \sum_{l=1}^{n} (X_i^k(t_{l+1}) - X_i^k(t_l))(X_j^m(t_{l+1}) - X_j^m(t_l)) = \int_{t_0}^{t} \Sigma_{i,j}^{k,m}(u)\, du$$

As before, the middle term of these equalities can be estimated with historical observations, while the last term is explicitly computable, in a slightly more complex way than the previous case. In fact, as done in the single commodity case, the best way to calculate it is via the polarization inequality (3), which leads us to calculate first $\left\langle X_i^k + X_j^m \right\rangle_t$.

Lemma 3.1 *We have*

$$\left\langle X_i^k + X_j^m \right\rangle_{T_{i,j}^0}^{T_{i,j}^1} = \int_{T_{i,j}^0}^{T_{i,j}^1} \left(\Sigma_{i,j}^{k,m}(t)\right)^2 dt = \int_{T_{i,j}^0}^{T_{i,j}^1} \Theta_{i,j}^{k,m} R^{k,m} \left(\Theta_{i,j}^{k,m}\right)^T dt$$

where

$$\Theta_{i,j}^{k,m} = \left(\begin{array}{cccc} e^{-\lambda^k(T_i - t)}\sigma_1^k, & \sigma_2^k, & e^{-\lambda^m(T_j - t)}\sigma_1^m, & \sigma_2^m \end{array} \right)$$

and

$$R^{k,m} = \left(\begin{array}{cc} \boldsymbol{\rho}^{k,k} & \boldsymbol{\rho}^{k,m} \\ \boldsymbol{\rho}^{m,k} & \boldsymbol{\rho}^{m,m} \end{array} \right) = \left(\begin{array}{cccc} 1 & \rho_{1,2}^{k,k} & \rho_{1,1}^{k,m} & \rho_{1,2}^{k,m} \\ \rho_{1,2}^{k,k} & 1 & \rho_{2,1}^{k,m} & \rho_{2,2}^{k,m} \\ \rho_{1,1}^{k,m} & \rho_{2,1}^{k,m} & 1 & \rho_{1,2}^{m,m} \\ \rho_{1,2}^{k,m} & \rho_{2,2}^{k,m} & \rho_{1,2}^{m,m} & 1 \end{array} \right)$$

Proof.

$$d\left(X_i^k + X_j^m\right) = \Theta_{i,j}^{k,m}\, dW^{k,m}(t) + \text{ drift} \tag{4}$$

where $W^{k,m}(t) := \left(W_1^k(t), W_2^k(t), W_1^m(t), W_2^m(t)\right)^T$ results in a Gaussian process with independent stationary increments, zero mean and self-correlation matrix given by $R^{k,m}$.

In order to calculate the quadratic variation of $X_i^k + X_j^m$, we now want to represent $W^{k,m}$ as a linear function of a 4-dimensional Brownian motion $\bar{W}^{k,m}$, i.e. $W^{k,m} = \Lambda^{k,m}\bar{W}^{k,m}$ (where the components of $\bar{W}^{k,m}$ are independent 1-dimensional Brownian motions), then we have $R^{k,m} = \Lambda^{k,m}(\Lambda^{k,m})^T$. We can choose to perform a Cholesky decomposition, so that $\Lambda^{k,m}$ can be taken as a lower triangular matrix: in fact, since $R^{k,m}$ is semipositive definite, it can be written as $R^{k,m} = L^{k,m}D^{k,m}\left(L^{k,m}\right)^T$, with $L^{k,m}$ unitary and lower triangular and $D^{k,m}$ diagonal; we can then let $\widetilde{\Lambda}^{k,m} := L^{k,m}\left(D^{k,m}\right)^{\frac{1}{2}}$, with $\left(D^{k,m}\right)^{\frac{1}{2}}$ the matrix having the diagonal elements which are square roots of those of $D^{k,m}$, we have that

$$\widetilde{\Lambda}^{k,m}\left(\widetilde{\Lambda}^{k,m}\right)^T = L^{k,m}\left(D^{k,m}\right)^{\frac{1}{2}}\left(L^{k,m}\left(D^{k,m}\right)^{\frac{1}{2}}\right)^T = L^{k,m}D^{k,m}\left(L^{k,m}\right)^T = R^{k,m}$$

Then,

$$d\left(X_i^k + X_j^m\right) = \Theta_{i,j}^{k,m}\tilde{\Lambda}^{k,m}d\bar{W}^{k,m} + \text{ drift}$$

so that

$$\left\langle X_i^k + X_j^m\right\rangle_{T_{i,j}^0}^{T_{i,j}^1} = \int_{T_{i,j}^0}^{T_{i,j}^1}\Theta_{i,j}^{k,m}R^{k,m}\left(\Theta_{i,j}^{k,m}\right)^T dt$$

□

Remark 3.1 Since the matrix $R^{k,m}$ has 6 free parameters, it turns out that also the lower triangular matrix $\tilde{\Lambda}^{k,m}$ has six free parameters. More in details, we have that $\tilde{\Lambda}_{11}^{k,m} = 1$, and the other three rows must have Euclidean norm equal to 1: counting the free parameters for each of the 4 rows, this confirms $0 + 1 + 2 + 3 = 6$ free parameters. However, recall that 2 out of 6 of the parameters of $R^{k,m}$ are already known from the results of the previous section, namely $\rho_{1,2}^{k,k} = \rho^k$ and $\rho_{1,2}^{k,k} = \rho^m$: this implies that also in $\tilde{\Lambda}^{k,m}$ we indeed have only 4 free parameters.

Remark 3.2 If one works with the six correlations of $R^{k,m}$, then one must impose that each correlation lies in $[-1,1]$ and that $R^{k,m}$ is semipositive definite, which is not easy to be verified. Instead, if one works with $\tilde{\Lambda}^{k,m}$, then one simply imposes the norm of each row being equal to 1: this implies that each entry satisfies $\tilde{\Lambda}_{ij}^{k,m} \in [-1,1]$, and by definition $R^{k,m}$ results to be semipositive definite, without the need of imposing it.

The integrand, in extended form, is given by

$$\begin{aligned}\Theta_{i,j}^{k,m}R^{k,m}\left(\Theta_{i,j}^{k,m}\right)^T \quad = \quad & \left(\sigma_1^k\right)^2 e^{-2\lambda^k(T_i-t)} + 2\left(\sigma_2^k\rho_{1,2}^{k,k} + \sigma_2^m\rho_{1,2}^{k,m}\right)\sigma_1^k e^{-\lambda^k(T_i-t)} + \\ & + \left(\sigma_1^m\right)^2 e^{-2\lambda^m(T_j-t)} + 2\left(\sigma_2^k\rho_{2,1}^{k,m} + \sigma_2^m\rho_{1,2}^{m,m}\right)\sigma_1^m e^{-\lambda^m(T_j-t)} + \\ & + 2\sigma_1^m\sigma_1^k\rho_{1,1}^{k,m}e^{-\lambda^kT_i-\lambda^mT_j}e^{\left(\lambda^k+\lambda^m\right)t} + 2\sigma_2^m\sigma_2^k\rho_{2,2}^{k,m} + \sigma_2^k + \sigma_2^m\end{aligned}$$

This results in

$$
\begin{aligned}
\left\langle X_i^k + X_j^m \right\rangle_{T_{i,j}^0}^{T_{i,j}^1} &= \frac{\left(\sigma_1^k\right)^2 \left(e^{-2\lambda^k\left(T_i - T_{i,j}^1\right)} - e^{-2\lambda^k\left(T_i - T_{i,j}^0\right)}\right)}{2\lambda^k} + \\
&+ \frac{2\left(\sigma_2^k \rho_{1,2}^{k,k} + \sigma_2^m \rho_{1,2}^{k,m}\right) \sigma_1^k \left(e^{-\lambda^k\left(T_i - T_{i,j}^1\right)} - e^{-\lambda^k\left(T_i - T_{i,j}^0\right)}\right)}{\lambda^k} + \\
&+ \frac{\left(\sigma_1^m\right)^2 \left(e^{-2\lambda^m\left(T_j - T_{i,j}^1\right)} - e^{-2\lambda^m\left(T_j - T_{i,j}^0\right)}\right)}{2\lambda^m} + \\
&+ \frac{2\left(\sigma_2^k \rho_{2,1}^{k,m} + \sigma_2^m \rho_{1,2}^{m,m}\right) \sigma_1^m \left(e^{-\lambda^m\left(T_j - T_{i,j}^1\right)} - e^{-\lambda^m\left(T_j - T_{i,j}^0\right)}\right)}{\lambda^m} + \\
&+ \frac{2\sigma_1^m \sigma_1^k \rho_{1,1}^{k,m} e^{-\lambda^k T_i - \lambda^m T_j} \left(e^{\left(\lambda^k + \lambda^m\right) T_{i,j}^1} - e^{\left(\lambda^k + \lambda^m\right) T_{i,j}^0}\right)}{\lambda^k + \lambda^m} + \\
&+ \left(2\sigma_2^m \sigma_2^k \rho_{2,2}^{k,m} + \sigma_2^k + \sigma_2^m\right) \left(T_{i,j}^1 - T_{i,j}^0\right)
\end{aligned}
$$

Plugging this into the polarization identity (3), we obtain

$$
\left\langle X_i^k, X_j^m \right\rangle_{T_{i,j}^0}^{T_{i,j}^1} = \rho_{1,2}^{k,m} A_{i,j}^{k,m} + \rho_{2,1}^{k,m} B_{i,j}^{k,m} + \rho_{1,1}^{k,m} C_{i,j}^{k,m} + \rho_{2,2}^{k,m} D^{k,m}
$$

where

$$
\begin{aligned}
A_{i,j}^{k,m} &:= \frac{\sigma_2^m \sigma_1^k \left(e^{-\lambda^k\left(T_i - T_{i,j}^1\right)} - e^{-\lambda^k\left(T_i - T_{i,j}^0\right)}\right)}{\lambda^k} \\
B_{i,j}^{k,m} &:= \frac{\sigma_2^k \sigma_1^m \left(e^{-\lambda^m\left(T_j - T_{i,j}^1\right)} - e^{-\lambda^m\left(T_j - T_{i,j}^0\right)}\right)}{\lambda^m} \\
C_{i,j}^{k,m} &:= \frac{\sigma_1^m \sigma_1^k \left(e^{-\lambda^k\left(T_i - T_{i,j}^1\right) - \lambda^m\left(T_j - T_{i,j}^1\right)} - e^{-\lambda^k\left(T_i - T_{i,j}^0\right) - \lambda^m\left(T_j - T_{i,j}^0\right)}\right)}{\lambda^k + \lambda^m} \\
D_{i,j}^{k,m} &:= \sigma_2^m \sigma_2^k \left(T_{i,j}^1 - T_{i,j}^0\right)
\end{aligned}
$$

are known from the calibration of the previous section, and the $\rho_{a,b}^{k,m}$ are still to be estimated. As before, one should aim to solve the linear system

$$
\begin{aligned}
\overline{\left\langle X_i^k, X_j^m \right\rangle_{T_{i,j}^0}^{T_{i,j}^1}} &= \rho_{1,2}^{k,m} A_{i,j}^{k,m} + \rho_{2,1}^{k,m} B_{i,j}^{k,m} + \rho_{1,1}^{k,m} C_{i,j}^{k,m} + \rho_{2,2}^{k,m} D_{i,j}^{k,m} \\
& \qquad \forall k, m \in \{1, \ldots, K\} \qquad \forall i \in N_k \qquad \forall j \in N_m
\end{aligned}
$$

which, as before, is overdetermined as soon as $|N_k| \times |N_m| > 4$. Thus, again we estimate the $\rho_{a,b}^{k,m}$ with a mean-square estimation, i.e. define the $\rho_{a,b}^{k,m}$ as the minimizers of the problem

$$
\min_{\rho_{a,b}^{k,m}} \sum_{i,jk,m} \left(\rho_{1,2}^{k,m} A_{i,j}^{k,m} + \rho_{2,1}^{k,m} B_{i,j}^{k,m} + \rho_{1,1}^{k,m} C_{i,j}^{k,m} + \rho_{2,2}^{k,m} D_{i,j}^{k,m} - \overline{\left\langle X_i^k, X_j^m \right\rangle_{T_{i,j}^0}^{T_{i,j}^1}} \right)^2
$$

Remark 3.3 Recall that $\rho_{a,b}^{k,m}$ can be expressed as functions of the entries of $\tilde{\Lambda}^{k,m}$: by Remark 3.2, this allows to not impose the positive semidefiniteness of the global correlation matrix $\boldsymbol{\rho}$, which is computationally very demanding.

4 An alternative calibration

Now we present an alternative calibration method, which is used among practitioners, but has the fault that, to be rigorous, works on an approximation of the original model. This method is based on the use of the so-called **rolling time series**. Assume from now on, as is quite realistic for those commodities which do not have forward contracts with long deliveries traded in the market, that the maturities $T_1,\dots,T_N$ are consecutive ends of months. Then the method of rolling time series consists in taking the forward contract with maturity month T_i and treating it, in the current month, as if its volatility were constant (and thus approximately equal to $\Sigma_k(s,T_i)$ with s a suitable point in the current month). When the current month ends and the next begins, take these observations and paste it to the observations of the forward with maturity month T_{i+1}: in this way, we obtain a time series of a forward contract with more or less the same relative maturity.

This method can be made rigorous by redefining the model as

$$\frac{dF^k(t,T)}{F^k(t,T)} = e^{-\frac{\lambda^k}{12}\lceil 12(T-t)\rceil}\sigma_1^k dW_1^k(t) + \sigma_2^k dW_2^k(t) \qquad \forall k = 1,\dots,K \tag{5}$$

If we, as before, denote $X_i^k(t) := \log F^k(t,T_i)$, then we have that

$$\bar{X}_i^k(t_1,t_2) := X_i^k(t_2) - X_i^k(t_1) = \int_{t_1}^{t_2} \sigma_1^k e^{-\frac{\lambda^k}{12}\lceil 12(T-s)\rceil} dW_1^k(s) + \int_{t_1}^{t_2} \sigma_2^k dW_2^k(s) + \text{ drift} \tag{6}$$

where "drift" denotes a quantity which is deterministic both under the risk-neutral probability as well as the real world probability (but of course possibly different). Thus, if we have an equispaced grid $t_1 < \ldots < t_\ell$, with $t_{l+1} - t_l \equiv \Delta$ in a given month, then $(\bar{X}_i^k(t_l,t_{l+1}))_{l=1,\dots,\ell-1}$ are i.i.d. Gaussian random variables with variance

$$\Sigma_{i,i}^{k,k} = \left(\sigma_1^k\right)^2 e^{-2\lambda^k(T_i)} \frac{e^{2\lambda^k\Delta}-1}{2\lambda^k} + 2\rho_{1,2}^{k,k}\sigma_1^k\sigma_2^k e^{-\lambda^k T_i}\frac{e^{\lambda^k\Delta}-1}{\lambda^k} + \left(\sigma_2^k\right)^2\Delta \tag{7}$$

and the same applies when we extend this to the rolling time series in the following months. Moreover, if we take two different maturities T_i, T_j, then the two sequences of Gaussian random variables $(\bar{X}_i^k(t_l,t_{l+1}))_{l=1,\dots,\ell-1}$ and $(\bar{X}_j^k(t_l,t_{l+1}))_{l=1,\dots,\ell-1}$ have covariance given by

$$\begin{aligned}
&\text{Cov}\,(\bar{X}_i^k(t_l,t_{l+1}), \bar{X}_j^k(t_l,t_{l+1})) = \Sigma_{i,j}^{k,k} := \\
&:= \left(\sigma_1^k\right)^2 e^{-\lambda^k\left(T_i+T_j\right)}\frac{e^{2\lambda^k t_2}-1}{2\lambda^k} + \sigma_1^k\sigma_2^k\rho_{1,2}^{k,k}e^{-\lambda^k T_i}\frac{e^{\lambda^k t_2}-1}{\lambda^k} + \\
&\quad +\sigma_2^k\sigma_1^k\rho_{1,2}^{k,k}e^{-\lambda^k T_j}\frac{e^{\lambda^k t_2}-1}{\lambda^k} + \left(\sigma_2^k\right)^2 t_2
\end{aligned} \tag{8}$$

These model variances and covariances can be estimated using the standard estimators

$$\bar{\Sigma}_{i,j}^{k,m} := s_{\bar{X}_i^k,\bar{X}_j^m} = \frac{\sum_l \bar{X}_i^k(t_l,t_{l+1})\bar{X}_j^m(t_l,t_{l+1})}{n} - \frac{\sum_l \bar{X}_i^k(t_l,t_{l+1})}{n}\frac{\sum_l \bar{X}_j^m(t_l,t_{l+1})}{n} \tag{9}$$

where n is the number of contemporary realizations of the time series $(\bar{X}_i^k(t_l,t_{l+1}))_l$ and $(\bar{X}_j^m(t_l,t_{l+1}))_l$. Define then $\bar{\mathbf{\Sigma}}^{k,m}$ as

$$\bar{\mathbf{\Sigma}}^{k,m} := \left(\bar{\Sigma}_{i,j}^{k,m}\right)_{i\leqslant N_k, j\leqslant N_m} = \begin{pmatrix} \bar{\Sigma}_{1,1}^{k,m} & \dots & \bar{\Sigma}_{1,N_m}^{k,m} \\ \vdots & \ddots & \vdots \\ \bar{\Sigma}_{N_k,1}^{k,m} & \dots & \bar{\Sigma}_{N_k,N_m}^{k,m} \end{pmatrix}$$

and $\bar{\mathbf{\Sigma}}$, which will be our realized covariance matrix, as

$$\bar{\mathbf{\Sigma}} \left(\ \bar{\mathbf{\Sigma}}^{k,m} \ \right)^{k,m\leqslant K} = \begin{pmatrix} \bar{\mathbf{\Sigma}}^{1,1} & \dots & \bar{\mathbf{\Sigma}}^{K,1} \\ \vdots & \ddots & \vdots \\ \bar{\mathbf{\Sigma}}^{1,K} & \dots & \bar{\mathbf{\Sigma}}^{K,K} \end{pmatrix}$$

This has to be compared to the model covariance matrix $\mathbf{\Sigma}$, defined as

$$\mathbf{\Sigma} \ := \ \left(\ \mathbf{\Sigma}^{k,m} \ \right)^{k,m\leqslant K} = \begin{pmatrix} \mathbf{\Sigma}^{1,1} & \dots & \mathbf{\Sigma}^{K,1} \\ \vdots & \ddots & \vdots \\ \mathbf{\Sigma}^{1,K} & \dots & \mathbf{\Sigma}^{K,K} \end{pmatrix}$$

where

$$\mathbf{\Sigma}^{k,m} := \left(\Sigma_{i,j}^{k,m}\left(p^{k,m}\right)\right)_{i\leqslant N_k, j\leqslant N_m} \quad = \quad \begin{pmatrix} \Sigma_{1,1}^{k,m} & \dots & \Sigma_{1,N_m}^{k,m} \\ \vdots & \ddots & \vdots \\ \Sigma_{N_k,1}^{k,m} & \dots & \Sigma_{N_k,N_m}^{k,m} \end{pmatrix}$$

As in the previous sections, one is tempted to let

$$\mathbf{\Sigma}\left(\boldsymbol{p}\right) = \bar{\mathbf{\Sigma}}$$

which is, as usual, overdetermined. We thus proceed as in the previous calibrations: first of all we estimate all the parameters for each commodity $k = 1, \dots, K$ separately, by making a least-square estimation in the usual way:

$$\min_{p^k} \sum_{i,j=1}^{N_k} \left(\Sigma_{i,j}^{k,k}\left(p^k\right) - \overline{\Sigma}_{i,j}^{k,k}\right)^2$$

Once that the $p^k = (\sigma_1^k, \sigma_2^k, \lambda^k, \rho^k)$ have been estimated, they are kept fixed and the second calibration is performed, again by least-squares, as

$$\min_{\rho_{a,b}^{k,m}} \sum_{k\neq m} \sum_{i=1}^{N_k} \sum_{j=1}^{N_m} \left(\Sigma_{i,j}^{k,m} - \overline{\Sigma}_{i,j}^{k,m}\right)^2$$

which gives the intercommodity correlations $\rho_{a,b}^{k,m}$, $a, b = 1, 2$, $k \neq m$.

Remark 4.1 As in Section 3, here too it is convenient to work with the Cholesky decomposition of the matrix $\mathbf{\Sigma}$: in this way, analogously with what happens in Remark 3.2, one has the same number of coefficients (in fact, $\mathbf{\Sigma}$ is symmetric and its Cholesky square root is lower triangular with the same dimension), but one has the constraint of $\mathbf{\Sigma}$ being positive semidefinite which is automatically satisfied.

5 Empirical findings

(work in progress)

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