Pricing and Hedging Quanto Options in Energy Markets

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Abstract

In energy markets, the use of quanto options have increased significantly in the recent years. The payoff from such options are typically triggered by an energy price and a measure of temperature and are thus suited for managing both price and volume risk in energy markets. Using an HJM approach we derive a closed form option pricing formula for energy quanto options, under the assumption that the underlying assets are log-normally distributed. Our approach encompasses several interesting cases, such as geometric Brownian motions and multi-factor spot models. We also derive delta and cross-gamma hedging parameters. Furthermore, we illustrate the use of our model by an empirical pricing exercise using NYMEX traded natural gas futures and CME traded HDD temperature futures for New York and Chicago

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1 Introduction

The market for standardized weather derivatives peaked in 2007 with a total volume of trades at the Chicago Mercantile Exchange (CME) close to 930,000 and a corresponding notional value of $17.9 billion[1]. Until recently these products have served as a tool for hedging volume risk of energy commodities like gas or power. Warm winters and cold summers lead to a decline in energy consumption as there is less need for heating respectively cooling. Cold winters and warm summers lead to a higher demand for energy, e.g., gas or electricity. However, in the last couple of years, this market has experienced severe retrenchment. In 2009, the total volume of trades dipped below 500,000, amounting to a notional value of around $5.3 billion. A big part of this sharp decline is attributed to the substantial increase in the market for tailor-made quantity-adjusting weather contracts (quanto contracts). Quanto deals with a size of $100 million have been reported. Market participants indicate that the demand for quanto-options are international with transactions being executed in the US, Europe, Australia and South America. The Weather Risk Management Association (WRMA) believes the developing market in India alone has a potential value of $2.35 billion in the next two or three years.

The label 'quanto options' have traditionally been assigned to a class of derivatives in currency markets used to hedge exposure to foreign currency risk. Although the same term is used for the specific type of energy options that we study in this paper, these two types of derivatives contracts are different. A typical *currency quanto option* have a regular call/put payoff structure, whereas the *energy quanto options* we study have a payoff structure similar to a product of call/put options. Pricing of currency quanto options have been extensively researched, and dates back to the original work of **?**. In comparison, research related to the pricing of quanto options in energy markets are scarce. Pricing options in energy markets are generally different from pricing options in financial markets since one has to take into account, *e.g.*, different asset dynamics, non-tradeable underlyings and less liquidity.

In energy markets, quanto options are mainly used to hedge exposure to both price and volume risk. This is contrary to industries with fixed prices over the short term, where hedging volumetric risks by using standardized weather derivatives is an appropriate hedging strategy. But when

[1]The numbers reported in this paragraph are taken directly from the article "A new direction for weather derivatives", published in the Energy Risk Magazine, June 2010.

earnings volatility is affected by more than one factor, the hedging problem quickly becomes more complex. Take as an example a gas distribution company which operates in an open wholesale market. Here it is possible to buy and sell within day or day ahead gas, and thus the company is exposed to movements in the spot price of gas as well as to variable volumes of sales due to the fluctuations in temperature. Their planned sales volumes per day and the price at which they are able to sell to their customers form the axes about which their exposure revolves. If for example, one of the winter months turn out to be warmer than usual, demand for gas would drop. This decline in demand would probably also affect the market price for gas, leading to a drop in gas price. The firm makes a loss against planned revenues equal to the short fall in demand multiplied by the difference between the retail price at which they would have sold if their customers had bought the gas and the market price where they must now sell their excess gas. The above example clearly illustrates that the adverse movements in market price and demand due to higher temperatures represent a kind of correlation risk which is difficult to properly hedge against. Using standard weather derivatives and, *e.g.*, futures contracts would most likely represent both an imperfect and rather expensive hedging strategy.

In order for quanto contracts to provide a superior risk management tool compared to standardized futures contracts, it is crucial that there is a significant correlation between the two underlying assets. In energy markets, payoffs of a quanto option is triggered by movements in both energy price and temperature (contracts). **?** document that temperature is important to forecast electricity prices and **?** document at strong relationship between natural gas prices and heating degree days (HDD).

The literature on energy quanto options is scarce. One exception is **?** who propose a bivariate time series model to capture the joint dynamics of energy prices and temperature. More specifically, they model the energy price and the average temperature using a sophisticated parameter-intensive econometric model. Since they aim to capture features like seasonality in means and variances, long memory, auto-regressive patterns and dynamic correlations, the complexity of their model leaves no other option than simulation based procedures to calculate prices. Moreover, they leave the issue of how one should hedge such options unanswered.

We also study the pricing of energy quanto options, but unlike **?** we derive analytical solutions to the option pricing problem. Such closed form solutions are easy to implement, fast to calculate and most importantly; they give a clear answer to how the energy quanto option should be properly hedged. Our idea is to convert the pricing problem by using futures contracts as underlying assets, rather than energy spot prices and temperature. We are able to do so since the typical energy quanto options have a payoff which can be represented as an "Asian" structure on the energy spot price and the temperature index. The markets for energy and weather organize futures with delivery periods, which will coincide with the aggregate or average spot price and temperature index at the end of the delivery period. Hence, any "Asian payoff" on the spot and temperature for a quanto option can be viewed as a "European payoff" on the corresponding futures contracts. It is this insight which is the key to our solution. This also gives the desirable feature that we can hedge the quanto option in terms of tradeable instruments, namely the underlying futures contracts. Note the contrast to viewing the energy quanto option as an "Asian-type" derivative on the energy spot and temperature index (cf. **?**). Temperature is not a tradeable asset, naturally, and in the case of power the spot is not as well. Thus, the hedging problem seems challenging in this context.

Using an HJM approach, we derive options prices under the assumption that futures dynamics are log-normally distributed with a possibly time-varying volatility. Furthermore, we explicitly derive delta- and cross-gamma hedging parameters. Our approach encompasses several models for the underlying futures prices, such as the standard bivariate geometric Brownian motion and the two-factor model proposed by **?**, and later extended by **?** to include seasonality. The latter model allows for time-varying volatility. We include an extensive empirical example to illustrate our findings. Using futures contracts on natural gas and HDD temperature index, we estimate relevant parameters in the seasonal two-factor model of **?** based on data collected from the New York Mercentile Exchange (NYMEX) and the Chicago Mercentile Exchange (CME). We compute prices for various energy quanto options and benchmark these against products of plain-vanilla European options on gas and HDD futures. The latter can be priced by the classical Black-76 formula (see **?**), and corresponds to the case of the energy quanto option for independent gas and temperature futures.

In section 2, we discuss the structure of energy quanto options as well as introduce the pricing problem. In section 3, we derive the pricing and hedging formulas and show how the model of **?** related to the general pricing formula. In section 4, we present the empirical case study and section 5 concludes.

2 Energy Quanto Options

In this section we first present typical examples of energy quanto options. We then argue that the pricing problem can be simplified using standardized futures contracts as the underlying assets.

2.1 Contract structure

Most energy quanto contracts have in common that payoffs are triggered by two underlying "assets"; temperature and energy price. Since these contracts are tailormade rather than standardized, the contract design varies. In its simplest form a quanto contract resembles a swap contract and has a payoff function S that looks like

$$S = Volume \times (T_{Var} - T_{Fix}) \times (P_{Var} - P_{Fix}) \tag{1}$$

Payoff is determined by the difference between some variable temperature measure (T_{Var}) and some fixed temperature measure (T_{Fix}), multiplied by the difference between variable and fixed energy price (P_{Var} and P_{Fix}). Note that the payoff might be negative, indicating that the buyer of the contract pays the required amount to the seller.

Entering into a swap contract of this type might be risky since the downside may potentially become large. For hedging purposes it seems more reasonable to buy a quanto structure with optionality, *i.e.*, so that you eliminate all downside risk. In Table 1 we show a typical example of how a quanto option might be structured. The example contract has a payoff which is triggered by an average gas price denoted E (defined as the average of daily prices for the last month), and it also offers an exposure to temperature through the accumulated number of Heating Degree Days (HDD) in the corresponding month (denoted H). The HDD index is commonly used as the underlying variable in temperature derivatives and is defined as

$$\sum_{t=\tau_1}^{\tau_2} \max(c - T_t, 0), \tag{2}$$

where c is some prespecified temperature threshold ($65°F$ or $18°C$), and T_t is the mean temperature on day t. If the number of HDDs H and the average gas price E is above the high strikes ($\overline{K}_I$ and $\overline{K}_E$ respectively), the owner of the option would receive a payment equal to the prespecified volume multiplied by the actual number of HDDs less the strike $\overline{K}_I$, multiplied by the difference between the average energy price less the strike price $\overline{K}_E$ (if $E > \overline{K}_E$). On the other hand, if it is warmer than usual and the number of HDDs dips below the lower strike of $\underline{K}_I$ and the energy price at the same time is lower than $\underline{K}_E$, the owner receives a payout equal to the volume multiplied by $\underline{K}_I$ less the actual number of HDDs multiplied by the difference between the strike price $\underline{K}_E$ and the average energy price. Note that the volume adjustment is varying between months, reflecting the fact that 'unusual' temperature changes might have a stronger impact on the optionholder's revenue in the coldest months like December and January. Also note that the price strikes may vary between months.

	Nov	Dec	Jan	Feb	Mar
(a) High Strike (HDDs)	$\overline{K}_I^{11}$	$\overline{K}_I^{12}$	$\overline{K}_I^{1}$	$\overline{K}_I^{2}$	$\overline{K}_I^{3}$
(b) Low Strike (HDDs)	$\underline{K}_I^{11}$	$\underline{K}_I^{12}$	$\underline{K}_I^{1}$	$\underline{K}_I^{2}$	$\underline{K}_I^{3}$
(a) High Strike (Price/mmBtu)	$\overline{K}_E^{11}$	$\overline{K}_E^{12}$	$\overline{K}_E^{1}$	$\overline{K}_E^{2}$	$\overline{K}_E^{3}$
(b) Low Strike (Price/mmBtu)	$\underline{K}_E^{11}$	$\underline{K}_E^{12}$	$\underline{K}_E^{1}$	$\underline{K}_E^{2}$	$\underline{K}_E^{3}$
Volume (mmBtu)	200	300	500	400	250

Table 1: A specification of a typical energy quanto option. The underlying process triggering payouts to the optionholder is accumulated number of heating-degree days H and monthly index gas price E. As an example the payoff for November will be: (a) In cold periods - $\max(H - \overline{K}_I, 0) \times \max(E - \overline{K}_E, 0) \times$ Volume. (b) In warm periods - $\max(\underline{K}_I - H, 0) \times \max(\underline{K}_E - E, 0) \times$ Volume. We see that the option pays out if both the underlying temperature and price variables exceed (dip below) the high strikes (low strikes).

This example illustrates why quanto options might be a good alternative to more standardized derivatives. The structure in the contracts takes into account the fact that extreme temperature variations might affect both demand and prices, and compensates the owner of the option by making payouts contingent on both prices and temperatures. The great possibility of tailoring these contracts provides the potential customers with a powerful and efficient hedging instrument.

2.2 Pricing Using Terminal Value of Futures

As described above energy quanto options have a payoff which is a function of two underlying assets; temperature and price. We focus on a class of energy quanto options which has a payoff function

$f(E, I)$, where E is an index of the energy price and I an index of temperature. To be more specific, we assume that the energy index E is given as the average spot price over some measurement period $[\tau_1, \tau_2]$, $\tau_1 < \tau_2$,

$$E = \frac{1}{\tau_2 - \tau_1} \sum_{u=\tau_1}^{\tau_2} S_u \,,$$

where S_u denotes the spot price of the energy. Furthermore, we assume that the temperature index is defined as

$$I = \sum_{u=\tau_1}^{\tau_2} g(T_u) \,,$$

for T_u being the temperature at time u and g some function. For example, if we want to consider a quanto option involving the HDD index, we choose $g(x) = \max(x - 18, 0)$. The quanto option is exercised at time τ_2, and its arbitrage-free price C_t at time $t \leq \tau_2$ is defined as by the following expression:

$$C_t = e^{-r(\tau_2 - t)} \mathbb{E}_t^{\mathbb{Q}} \left[f \left(\frac{1}{\tau_2 - \tau_1} \sum_{u=\tau_1}^{\tau_2} S_u, \sum_{u=\tau_1}^{\tau_2} g(T_u) \right) \right] \,. \tag{3}$$

Here, $r > 0$ denotes the risk-free interest rate, which we for simplicity assumes constant. The pricing measure is denoted $\mathbb{Q}$, and $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ is the expectation operator with respect to $\mathbb{Q}$, conditioned on the market information at time t given by the filtration $\mathcal{F}_t$.

We now argue how to relate the price of the quanto option to futures contracts on the energy and temperature indices E and I. Observe that the price at time $t \leq \tau_2$ of a futures contract written on some energy price, e.g, natural gas, with delivery period $[\tau_1, \tau_2]$ is given by

$$F_t^E(\tau_1, \tau_2) = \mathbb{E}_t^{\mathbb{Q}} \left[\frac{1}{\tau_2 - \tau_1} \sum_{u=\tau_1}^{\tau_2} S_u \right] \,.$$

At time $t = \tau_2$, we find from the conditional expectation that

$$F_{\tau_2}^E(\tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \sum_{u=\tau_1}^{\tau_2} S_u \,,$$

i.e., the futures price is exactly equal to what is being delivered. Applying the same argument to a futures written on the temperature index, with price dynamics denoted $F_t^I(\tau_1, \tau_2)$, we immediately see that the following must be true for the quanto option price:

$$
\begin{aligned}
C_t &= e^{-r(\tau_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[f\left(\frac{1}{\tau_2-\tau_1}\sum_{u=\tau_1}^{\tau_2} S_u, \sum_{u=\tau_1}^{\tau_2} g(T_u)\right)\right] \\
&= e^{-r(\tau_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[f\left(F_{\tau_2}^{E}(\tau_1,\tau_2), F_{\tau_2}^{I}(\tau_1,\tau_2)\right)\right] . \qquad (4)
\end{aligned}
$$

Equation (4) shows that the price of a quanto option with payoff being a function of the energy index E and temperature index I must be the same as if the payoff was a function of the terminal values of two futures contracts written on the energy and temperature indices, and with the delivery period being equal to the contract period specified by the quanto option. Hence, we view the quanto option as an option written on the two futures contracts rather than on the two indices. This is advantageous from the point of view that the futures are traded financial assets. We note in passing that we may extend the above argument to quanto options where the measurement periods of the energy and the temperature indices are not the same.

To compute the price in (4) we must have a model for the futures price dynamics $F_t^E(\tau_1,\tau_2)$ and $F_t^I(\tau_1,\tau_2)$. The dynamics must account for the dependency between the two futures, as well as their marginal behavior. The pricing of the energy quanto option has thus been transferred from modeling the joint spot energy and temperature dynamics followed by computing the $\mathbb{Q}$-expectation of an index of these, to modeling the joint futures dynamics and pricing a European-type option on these. The former approach is similar to pricing an Asian option, which for most relevant models and cases is a highly difficult task. Remark also that by modeling and estimating the futures dynamics to market data, we can easily obtain the market-implied pricing measure $\mathbb{Q}$. We will see this in practice in Section 4 where we analyze the case of gas and HDD futures. If one chooses to model the underlying energy spot prices and temperature dynamics, one obtains a dynamics under the market probability $\mathbb{P}$, and not under the pricing measure $\mathbb{Q}$. Additional hypotheses must be made in the model to obtain this. Moreover, for most interesting cases the quanto option must be priced by Monte Carlo or some other computationally demanding method (see **?**). Finally, but not less importantly, with the representation in (4) at hand one can discuss the issue of hedging energy quanto options in terms of the underlying futures contracts.

In many energy markets, the futures contracts are not traded within their delivery period. That means that we can only use the market for futures up to time τ_1. This has a clear consequence on the possibility to hedge these contracts, as a hedging strategy inevitably will be a continuously rebalanced portfolio of the futures up to the exercise time τ_2. As this is possible to perform only up to time τ_1 in many markets, we face an incomplete market situation where the quanto option cannot be hedged perfectly. Moreover, it is to be expected that the dynamics of the futures price have different characteristics within the delivery period than prior to start of delivery, if it can be traded for times $t \in (\tau_1, \tau_2]$. The reason being that we have less uncertainty as the remaining delivery period of the futures become shorter. The entry time of such a contract is most naturally taking place prior to delivery period. However, for marking-to-market purposes, one is interested in the price C_t also for $t \in (\tau_1, \tau_2]$. The issuer of the quanto option may be interested in hedging the exposure, and therefore also be concerned of the behavior of prices within the delivery period.

Before we start looking into the details of pricing quanto options we investigate the option contract of the type described in section 2.1 in more detail. This contract covers a period of 5 months, from November through March. Since this contract essentially is a sum of one-period contracts we focus our attention on such, *i.e.*, an option covering only one month of delivery period $[\tau_1, \tau_2]$. Recall that the payoff in the contract is a function of some average energy price and accumulated number of HDDs. From the discussion in the previous section we know that rather than using spot price and HDD as underlying assets, we can instead use the terminal value of futures contracts written on price and HDD, respectively. The payoff function $p(F^E_{\tau_2}(\tau_1, \tau_2), F^I_{\tau_2}(\tau_1, \tau_2), \overline{K}_E, \overline{K}_I, \underline{K}_E, \underline{K}_I) = p$ of this quanto contract is defined as

$$\begin{aligned} p = \gamma \times \big[& \max\left(F^E_{\tau_2}(\tau_1, \tau_2) - \overline{K}_E, 0\right) \times \max\left(F^I_{\tau_2}(\tau_1, \tau_2) - \overline{K}_I, 0\right) \\ & + \max\left(\underline{K}_E - F^E_{\tau_2}(\tau_1, \tau_2), 0\right) \times \max\left(\underline{K}_I - F^I_{\tau_2}(\tau_1, \tau_2), 0\right)\big] \ , \end{aligned} \tag{5}$$

where γ is the contractual volume adjustment factor. Note that the payoff function in this contract consists of two parts, the first taking care of the situation where temperatures are colder than usual (and prices higher than usual), and the second taking care of the situation where temperatures are warmer than usual (and prices lower than usual). The first part is a product of two call options, whereas the second part is a product of two put options. To illustrate our pricing approach in the simplest possible way it suffices to look at the product call structure with the volume adjuster γ

normalized to 1, *i.e.*, we want to price an option with the following payoff function:

$$\hat{p}\left(F^E_{\tau_2}(\tau_1,\tau_2), F^I_{\tau_2}(\tau_1,\tau_2), \overline{K}_E, \overline{K}_I\right) = \max\left(F^E_{\tau_2}(\tau_1,\tau_2) - \overline{K}_E, 0\right) \times \max\left(F^I_{\tau_2}(\tau_1,\tau_2) - \overline{K}_I, 0\right). \quad (6)$$

In the remaining part of this paper we will focus on this particular choice of a payoff function for the energy quanto option. It corresponds to choosing the function f as $f(E,I) = \max(E - \overline{K}_E, 0) \times \max(I - \overline{K}_I, 0)$ in (4). Other combinations of put-call mixes as well as different delivery periods for the energy and temperature futures can easily be studied by a simple modification of what comes.

3 Pricing and hedging an energy quanto option

Suppose that the two futures price dynamics under the pricing measure $\mathbb{Q}$ can be expressed as

$$F^E_T(\tau_1,\tau_2) = F^E_t(\tau_1,\tau_2)\exp(\mu_E + X) \quad (7)$$

$$F^I_T(\tau_1,\tau_2) = F^I_t(\tau_1,\tau_2)\exp(\mu_I + Y) \quad (8)$$

where $t \leq T \leq \tau_2$, and X, Y are two random variables independent of $\mathcal{F}_t$, but depending on t, T, τ_1 and τ_2. We suppose that (X,Y) is a bivariate normally distributed random variable with mean zero, with covariance structure depending on t, T and τ_2. We denote $\sigma^2_X = Var(X)$, $\sigma^2_Y = Var(Y)$ and $\rho_{X,Y} = corr(X,Y)$. Obviously, σ_X, σ_Y and $\rho_{X,Y}$ are depending on t, T, τ_1 and τ_2. Moreover, as the futures price naturally is a martingale under the pricing measure $\mathbb{Q}$, we have $\mu_E = -\sigma^2_X/2$ and $\mu_I = -\sigma^2_Y/2$.

Our general representation of the futures price dynamics (7) and (8) encompasses many interesting models. For example, a bivariate geometric Brownian motion looks like

$$F^E_T(\tau_1,\tau_2) = F^E_t(\tau_1,\tau_2)\exp\left(-\frac{1}{2}\sigma^2_E(T-t) + \sigma_E(W_T - W_t)\right)$$

$$F^I_T(\tau_1,\tau_2) = F^I_t(\tau_1,\tau_2)\exp\left(-\frac{1}{2}\sigma^2_I(T-t) + \sigma_I(B_T - B_t)\right)$$

with two Brownian motions W and B being correlated. We can easily associate this GBM to the general set-up above by setting $\mu_E = -\sigma^2_E(T-t)/2$, $\mu_I = -\sigma^2_I(T-t)/2$, $\sigma_X = \sigma_E\sqrt{T-t}$, $\sigma_Y = \sigma_I\sqrt{T-t}$, and $\rho_{X,Y}$ being the correlation between the two Brownian motions. In section 3.3

we show that also the two-factor model by **?** and the later extension of **?** fits this framework.

3.1 A General Solution

The price of the quanto option at time t is

$$C_t = e^{-r(\tau_2 - t)} \mathbb{E}_t^{\mathbb{Q}} \left[\hat{p} \left(F_{\tau_2}^E(\tau_1, \tau_2), F_{\tau_2}^I(\tau_1, \tau_2), \overline{K}_E, \overline{K}_I \right) \right] , \tag{9}$$

where the notation $\mathbb{E}^{\mathbb{Q}}$ states that the expectation is taken under the pricing measure $\mathbb{Q}$. Given these assumptions Proposition 1 below states the closed-form solution of the energy quanto option.

Proposition 1. *For two assets following the dynamics given by (7) and (8), the time t market price of an European energy quanto option with exercise at time τ_2 and payoff described by (6) is given by*

$$\begin{aligned} C_t = e^{-r(\tau_2 - t)} \Bigg(& F_t^E(\tau_1, \tau_2) F_t^I(\tau_1, \tau_2) e^{\rho_{X,Y}\sigma_X\sigma_Y} M(y_1^{***}, y_2^{***}; \rho_{X,Y}) - F_t^E(\tau_1, \tau_2) \overline{K}_I M\left(y_1^{**}, y_2^{**}; \rho_{X,Y}\right) \\ & - F_t^I(\tau_1, \tau_2) \overline{K}_E M\left(y_1^{*}, y_2^{*}; \rho_{X,Y}\right) + \overline{K}_E \overline{K}_I M\left(y_1, y_2; \rho_{X,Y}\right) \Bigg) \end{aligned}$$

where

$$\begin{aligned} y_1 &= \frac{\log(F_t^E(\tau_1, \tau_2)) - \log(\overline{K}_E) - \frac{1}{2}\sigma_X^2}{\sigma_X} , & y_2 &= \frac{\log(F_t^I(\tau_1, \tau_2)) - \log(\overline{K}_I) - \frac{1}{2}\sigma_Y^2}{\sigma_Y} , \\ y_1^* &= y_1 + \rho_{X,Y}\sigma_Y , & y_2^* &= y_2 + \sigma_Y , \\ y_1^{**} &= y_1 + \sigma_X , & y_2^{**} &= y_2 + \rho_{X,Y}\sigma_X , \\ y_1^{***} &= y_1 + \rho_{X,Y}\sigma_Y + \sigma_X , & y_2^{***} &= y_2 + \rho_{X,Y}\sigma_X + \sigma_Y . \end{aligned}$$

Here $M(x, y; \rho)$ denotes the standard bivariate normal cumulative distribution function with correlation ρ.

Proof. Observe that the payoff function in (6) can be rewritten in the following way:

$$\begin{aligned}\hat{p}(F^E, F^I, \overline{K}_E, \overline{K}_I) &= \max(F^E - \overline{K}_E, 0) \cdot \max(F^I - \overline{K}_I, 0)\\ &= \left(F^E - \overline{K}_E\right) \cdot \left(F^I - \overline{K}_I\right) \cdot 1_{\{F^E > \overline{K}_E\}} \cdot 1_{\{F^I > \overline{K}_I\}}\\ &= F^E F^I \cdot 1_{\{F^E > \overline{K}_E\}} \cdot 1_{\{F^I > \overline{K}_I\}} - F^E \overline{K}_I \cdot 1_{\{F^E > \overline{K}_E\}} \cdot 1_{\{F^I > \overline{K}_I\}}\\ &- F^I \overline{K}_E \cdot 1_{\{F^E > \overline{K}_E\}} \cdot 1_{\{F^I > \overline{K}_I\}} + \overline{K}_E \overline{K}_I \cdot 1_{\{F^E > \overline{K}_E\}} \cdot 1_{\{F^I > \overline{K}_I\}}.\end{aligned}$$

The problem of finding the market price of the European quanto option is thus equivalent to the problem of calculating the expectations under the pricing measure $\mathbb{Q}$ of the four terms above. The four expectations are derived in Appendix C in details. □

3.2 Hedging

Based on the formula given in Proposition 1 we derive the delta and cross-gamma hedging parameters, which are straightforwardly calculated from partial differentiation of the price C_t with respect to the futures prices. All hedging parameters are given by the current futures price of the two underlying contracts and are therefore simple to implement in practice. The delta hedge with respect to the energy futures is given by

$$\begin{aligned}\frac{\partial C_t}{\partial F_t^E(\tau_1,\tau_2)} &= F_t^I(\tau_1,\tau_2) e^{-r(\tau_2-t)+\rho_{X,Y}\sigma_X\sigma_Y} \left(M\left(y_1^{***}, y_2^{***}; \rho_{X,Y}\right) + B(y_1^{***}) N(y_2^{***} - \rho_{X,Y}) \frac{1}{\sigma_X} \right)\\ &- \overline{K}_I e^{-r(\tau_2-t)} \left(M\left(y_1^{**}, y_2^{**}; \rho_{X,Y}\right) + B(y_1^{**}) N(y_2^{**} - \rho_{X,Y}) \frac{1}{\sigma_X} \right)\\ &- \frac{F_t^I(\tau_1,\tau_2)\overline{K}_E}{F_t^E(\tau_1,\tau_2)\sigma_X} e^{-r(\tau_2-t)} B(y_1^*) N(y_2^* - \rho_{X,Y})\\ &+ \frac{\overline{K}_E\overline{K}_I}{F_t^E(\tau_1,\tau_2)\sigma_X} e^{-r(\tau_2-t)} B(y_1) N(y_2 - \rho_{X,Y}), \qquad (10)\end{aligned}$$

where $N(\cdot)$ denotes the standard normal cumulative distribution function, and

$$B(x) = \frac{e^{\left(x^2 - \rho_{X,Y}^2\right)}}{4\pi^2 \left(1 - \rho_{X,Y}^2\right)}.$$

The delta hedge with respect to the temperature index futures is of course analogous to the energy delta hedge, only with the substitutions $F_t^E(\tau_1,\tau_2) = F_t^I(\tau_1,\tau_2)$, $y_1^{***} = y_2^{***}$, $y_1^{**} = y_2^{**}$, $y_1^* = y_2^*$,

$y_1 = y_2$, $\sigma_Y = \sigma_X$ and $\sigma_X = \sigma_Y$. The cross-gamma hedge is given by

$$
\begin{aligned}
&\frac{\partial C_t^2}{\partial F_t^E(\tau_1,\tau_2)\partial F_t^I(\tau_1,\tau_2)} \\
&\quad = e^{-r(\tau_2-t)+\rho_{X,Y}\sigma_X\sigma_Y}\left(M\left(y_1^{***}, y_2^{***}; \rho_{X,Y}\right) + B(y_2^{***})N(y_1^{***}-\rho_{X,Y})\frac{1}{\sigma_Y}\right) \\
&\qquad + e^{-r(\tau_2-t)+\rho_{X,Y}\sigma_X\sigma_Y} B(y_1^{***})\left(N(y_2^{***}-\rho_{X,Y})\frac{1}{\sigma_X} + n(y_2^{***}-\rho_{X,Y})\frac{1}{\sigma_Y}\right) \\
&\qquad - \frac{\overline{K}_I}{F_t^I(\tau_1,\tau_2)\sigma_Y}e^{-r(\tau_2-t)}\left(B(y_2^{**})N(y_1^{**}-\rho_{X,Y}) + B(y_1^{**})n(y_2^{**}-\rho_{X,Y})\frac{1}{\sigma_X}\right) \\
&\qquad - \frac{\overline{K}_E}{F_t^E(\tau_1,\tau_2)\sigma_X}e^{-r(\tau_2-t)}B(y_1^{*})\left(N(y_2^{*}-\rho_{X,Y}) + n(y_2^{*}-\rho_{X,Y})\frac{1}{\sigma_Y}\right) \\
&\qquad + \frac{\overline{K}_E\overline{K}_I}{F_t^E(\tau_1,\tau_2)F_t^I(\tau_1,\tau_2)(\sigma_X+\sigma_Y)}e^{-r(\tau_2-t)}B(y_1)n(y_2-\rho_{X,Y})\,, \qquad (11)
\end{aligned}
$$

where $n(\cdot)$ denotes the standard normal probability density function. In our model it is possible to hedge the quanto option perfectly, with positions described above by the three delta and gamma parameters. In practice, however, this would be difficult due to low liquidity in for example the temperature market. Furthermore, as discussed in Section 2.2, we cannot in all markets trade futures within the delivery period, putting additional restrictions on the suitability of the hedge. In such cases, the parameters above will guide in a partial hedging of the option.

3.3 Two-dimensional Schwartz-Smith Model with Seasonality

The popular commodity price model proposed by **?** is a natural starting point for deriving dynamics of energy futures. In this model the log-spot price is the sum of two processes, one representing the long term dynamics of the commodity prices in form of an arithmetic Brownian motion and one representing the short term deviations from the long run dynamics in the form of an Ornstein-Uhlenbeck process with a mean reversion level of zero. As we have mentioned already, **?** extends the model of **?** to include seasonality. The dynamics under $\mathbb{P}$ is given by

$$
\begin{aligned}
\log S_t &= \Lambda(t) + X_t + Z_t\,, \\
dX_t &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma d\widetilde{W}_t\,, \\
dZ_t &= -\kappa Z_t dt + \nu d\widetilde{B}_t\,.
\end{aligned}
$$

Here $\widetilde{B}$ and $\widetilde{W}$ are correlated Brownian motions and μ, σ, κ and η are constants. The deterministic function $\Lambda(t)$ describes the seasonality of the log-spot prices. In order to price a futures contract written on an underlying asset with the above dynamics, a measure change from $\mathbb{P}$ to an equivalent probability $\mathbb{Q}$ is made:

$$
\begin{aligned}
dX_t &= \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \\
dZ_t &= -\left(\lambda_Z + \kappa Z_t\right) dt + \nu dB_t^i.
\end{aligned}
$$

Here, $\alpha = \mu - \lambda_X$, and λ_X and λ_Z are constant market prices of risk associated with X_t and Z_t for asset i, respectively. This correponds to a Girsanov transform of $\widetilde{B}$ and $\widetilde{W}$ by a constant drift so that B and W become two correlated $\mathbb{Q}$-Brownian motions. As is well-known for the Girsanov transform, the correlation between B and W is the same under $\mathbb{Q}$ as the one for $\widetilde{B}$ and $\widetilde{W}$ under $\mathbb{P}$ (see **?**). As it follows from **?**, the futures price $F_t(\tau)$ at time $t \geq 0$ of a contract with delivery at time $\tau \geq t$ has the following form on log-scale (note that it is the Schwartz-Smith futures prices scaled by a seasonality function):

$$
\log F_t(\tau) = \Lambda(\tau) + A(\tau - t) + X_t + Z_t e^{-\kappa(\tau - t)}\,, \tag{12}
$$

where $A(\tau) = \alpha\tau - \frac{\lambda_Z - \rho\sigma\nu}{\kappa}\left(1 - e^{-\kappa\tau}\right) + \frac{\nu^2}{4\kappa}\left(1 - e^{-2\kappa\tau}\right)$. The futures prices are affine in the two factors X and Z driving the spot price and scaled by functions of time to delivery $\tau - t$. **?** chooses to parametrize the seasonality function Λ by a linear combination of cosine and sine functions:

$$
\Lambda(t) = \sum_{k=1}^{K} \left(\gamma_k \cos(2\pi k t) + \gamma_k^* \sin(2\pi k t)\right) \tag{13}
$$

However, other choices may of course be made to match price observations in the market in question.

In this paper we have promoted the fact that the payoff of energy quanto options can be expressed in terms of the futures prices of energy and temperature index. One may use the above procedure to derive futures price dynamics from a model of the spot. However, one may also state directly a futures price dynamics in the fashion of Heath-Jarrow-Morton (HJM). The HJM approach has been proposed to model energy futures by **?**, and later investigated in detail by **?** (see also **?** and **?**). We follow this approach here, proposing a joint model for the energy and temperature index futures price based on the above seasonal Schwartz-Smith model.

In stating such a model, we must account for the fact that the futures in question are delivering over a period $[\tau_1, \tau_2]$, and *not* at a fixed delivery time τ. There are many ways to overcome this obstacle. For example, as suggested by **?**, one can model $F_t(\tau)$ and define the futures price $F_t(\tau_1, \tau_2)$ of a contract with delivery over $[\tau_1, \tau_2]$ as

$$F_t(\tau_1, \tau_2) = \sum_{u=\tau_1}^{\tau_2} F_t(u)\,.$$

If the futures price $F_t(\tau_1, \tau_2)$ refers to the average of the spot, we naturally divide by the number of times delivery takes place in the relation above. In the case of exponential models, as we consider, this leads to expressions which are not analytically tractable. Another, more attractive alternative, is to let $F_t(\tau_1, \tau_2)$ itself follow a dynamics of the form (12) with some appropriately chosen dependency on τ_1 and τ_2. For example, we may choose $\tau = \tau_1$ in (12), or $\tau = (\tau_1 + \tau_2)/2$, or any other time within the delivery period $[\tau_1, \tau_2]$. In this way, we will account for the delivery time-effect in the futures price dynamics, sometimes referred to as the Samuelson effect. We remark that it is well-known that for futures delivering over a period, the volatility will not converge to that of the underlying spot as time to delivery goes to zero (see **?**). By the above choices, we obtain namely that effect. Note that the futures price dynamics will not be defined for times t after the "delivery" τ. Hence, if we choose $\tau = \tau_1$, we will only have a futures price lasting up to time $t \leq \tau_1$, and left undefined thereafter.

In order to jointly model the energy and temperature futures price, two futures dynamics of the type in (12) are connected by allowing the Brownian motions to be correlated across assets. We will have four Brownian motions W^E, B^E, W^I and B^I in our two-asset two-factor model. These are assumed correlated as follows: $\rho_E = corr(W_1^E, B_1^E)$, $\rho_I = corr(W_1^I, B_1^I)$, $\rho_W = corr(W_1^E, W_1^I)$ and $\rho_B = corr(B_1^E, B_1^I)$. Moreover, we have cross-correlations given by

$$\rho_{I,E}^{W,B} = corr(W_1^I, B_1^E)\,,$$
$$\rho_{E,I}^{W,B} = corr(W_1^E, B_1^I)\,.$$

We refer to Appendix A for an explicit construction of four such correlated Brownian motions from four independent ones. In a HJM-style, we assume that the joint dynamics of the futures price processes $F_t^E(\tau_1,\tau_2)$ and $F_t^I(\tau_1,\tau_2)$ under $\mathbb{Q}$ is given by

$$\frac{dF_t^i(\tau_1,\tau_2)}{F_t^i(\tau_1,\tau_2)} = \sigma_i dW_t^i + \eta_i(t) dB_t^i\,, \tag{14}$$

for $i = E, I$ and with

$$\eta_i(t) = \nu_i e^{-\kappa^i(\tau_2 - t)}\,. \tag{15}$$

Note that we suppose that the futures price is a martingale with respect to the pricing measure $\mathbb{Q}$, which is natural from the point of view that we want an arbitrage-free model. Moreover, we have made the explicit choice here that $\tau = \tau_2$ in (12) when modelling the delivery time effect.

Note that

$$d\log F_t^i(\tau_1,\tau_2) = -\frac{1}{2}\left(\sigma_i^2 + \eta_i(t)^2 + 2\rho_i\sigma_i\eta_i(t)\right)dt + \sigma_i d\tilde{W}_t^i + \eta_i(t) d\tilde{B}_t^i$$

for $i = E, I$. Hence, we can make the representation $F_T^E(\tau_1,\tau_2) = F_t^E(\tau_1,\tau_2)\exp\left(-\mu_E + X\right)$ by choosing

$$X \sim \mathcal{N}\left(0, \underbrace{\int_t^T \left(\sigma_E^2 + \eta_E(s)^2 + 2\rho_E\sigma_E\eta_E(s)\right)ds}_{\sigma_X^2}\right), \qquad \mu_E = -\frac{1}{2}\sigma_X^2$$

and similar for $F_T^I(\tau_1,\tau_2)$. These integrals can be computed analytically in the above model, where $\eta_i(t) = \nu_i e^{-\kappa^i(\tau_2 - t)}$. We can also compute the correlation $\rho_{X,Y}$ analytically, since $\rho_{X,Y} = \frac{cov(X,Y)}{\sigma_X\sigma_Y}$ and

$$cov(X,Y) = \rho_W\int_t^T \sigma_E\sigma_I ds + \rho_{E,I}^{W,B}\int_t^T \sigma_E\eta_I(s)ds + \rho_{I,E}^{W,B}\int_t^T \eta_E(s)\sigma_I ds + \rho_B\int_t^T \eta_E(s)\eta_I(s)ds\,.$$

A closed-form expression of this covariance can be computed. In the special case of zero cross-correlations this simplifies to

$$cov(X,Y) = \rho_W\int_t^T \sigma_E\sigma_I ds + \rho_B\int_t^T \eta_E(s)\eta_I(s)ds$$

The exact expressions for σ_X, σ_Y and $cov(X,Y)$ in the two-dimensional Schwartz-Smith model with seasonality are presented in Appendix D.

This bivariate futures price model has a form that can be immediately used for pricing energy quanto options by inferring the result in Proposition 1. We shall come back to this model in the empirical case study in Section 4. We note that our pricing approach only looks at futures dynamics up to the start of the delivery period τ_1. As briefly discussed in Section 2.2 it is reasonable to expect that the dynamics of a futures contract should be different within the delivery period $[\tau_1, \tau_2]$. For times t within $[\tau_1, \tau_2]$ we will in the case of the energy futures have

$$F_t(\tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \sum_{u=\tau_1}^{t} S_u + \mathbb{E}_t^{\mathbb{Q}} \left[\frac{1}{\tau_2 - \tau_1} \sum_{u=t+1}^{\tau_2} S_u \right].$$

Thus, the futures price must consist of two parts, the first simply the tracked observed energy spot up to time t, and next the current futures price of a contract with delivery period $[t, \tau_2]$. This latter part will have a volatility that must go to zero as t tends to τ_2.

4 Empirical Analysis

In this section, we present an empirical study of energy quanto options written on natural gas and HDD temperature index. We present the futures price data which consitute the basis of our analysis, and next estimate the parameters in the joint futures price model (14). We then discuss the impact of correlation on the valuation of the option to be priced.

4.1 Data

A futures contract on Heating Degree Days are traded on CME for several cities for the months October, November, December, January, February, March, and April a couple of years out. The contract value is 20$ for each HDD throughout the month and it trades until the beginning of the concurrent month. The underlying is one month of accumulated HDD's for a specific location. The futures price is denoted by $F_t^I(\tau_1, \tau_2)$ and settled on the index $\sum_{u=\tau_1}^{\tau_2} HDD_u$. We observe the futures prices for seven specific combinations of $[\tau_1, \tau_2]$'s per year. We let the futures price follow a price process of the type (12) discussed in the previous Subsection.

For liquidity reasons, we do not include all data. Liquidity is limited after the first year, so for every day we choose the first seven contracts, where the index period haven't started yet. I.e., for

January 2nd, 2007, we use the February 2007, March 2007, April 2007, October 2007, November 2007, December 2007 and January 2008 contracts. The choice of the seasonality function Λ is copied from **?**, *i.e.*, $K = 1$ in equation 13, that is, a sum of a sine and cosine function with yearly frequency.

The chosen locations are New York and Chicago, since these are located in an area with fairly large gas consumption. The development in the futures curves are shown in Figures 1 and 2.

Futures contracts for delivery of gas is traded on NYMEX for each month ten years out. The underlying is delivery of gas throughout a month and the price is per unit. The contract trades until a couple of days before the delivery month. Many contracts are closed prior to the last trading day, we choose the first 12 contracts for delivery at least one month later. I.e., for January 2nd, we use March 2007 to February 2008 contracts. The choice of Λ is also in this case borrowed from **?**, where we for this case choose $K = 2$ in equation 13. The evolution of the futures gas curves is shown in Figure 3.

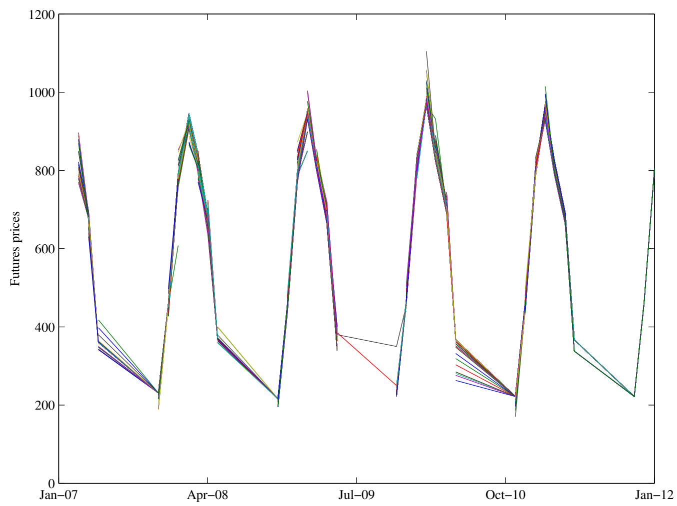

Figure 1: The evolution of the HDD futures curve for New York as a function of time of maturity. For each day t, $F_t(\tau_1^i, \tau_2^i)$ as a function of τ_2 where index i represents the next seven contracts maturing.

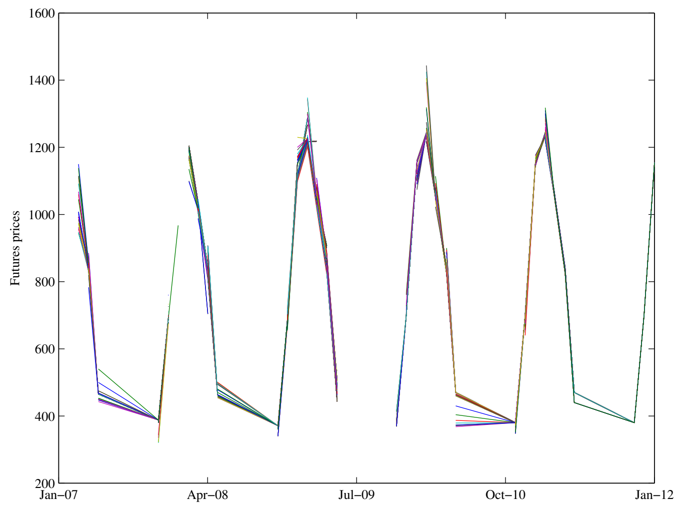

Figure 2: The evolution of the HDD futures curve for Chicago as a function of time of maturity. For each day t, $F_t(\tau_1^i, \tau_2^i)$ as a function of τ_2 where index i represents the next seven contracts maturing.

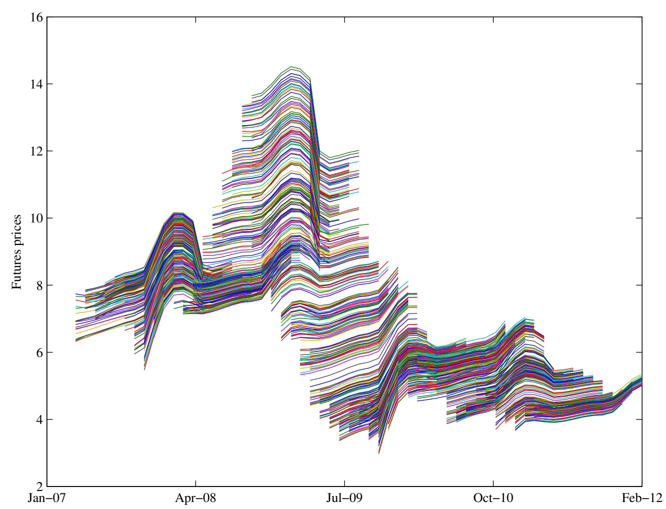

Figure 3: The evolution of the futures curve for Natural Gas as a function of time of maturity. For each day t, $F_t(\tau_1^i, \tau_2^i)$ as a function of τ_2 where index i represents the next 12 contracts maturing.

4.2 Estimation Results

We estimate the parameters using Kalman filter teqniques (see Appendix F). The resulting parameter estimates are reported in Table 2 with standard errors based on the Hessian of the log-likelihood function given in parentheses. Figures 4-5 show the the model fit along with observed data and

Figures 6-9 show RMSE plots.

	New York	Gas	Chicago	Gas
κ	16.5654 (1.1023)	0.6116 (0.0320)	18.8812 (1.3977)	0.6034 (0.0317)
σ	0.0494 (0.0059)	0.2342 (0.0200)	0.0379 (0.0051)	0.2402 (0.0209)
ν	3.6517 (0.6197)	0.6531 (0.0332)	4.3980 (0.8908)	0.6647 (0.0335)
ρ	-0.6066 (0.0801)	-0.6803 (0.0656)	-0.5509 (0.0948)	-0.7038 (0.0611)
σ_ϵ	0.0655 (0.0006)	0.0199 (0.0001)	0.0554 (0.0005)	0.0199 (0.0001)
γ_1	0.9044 (0.0023)	0.0500 (0.0003)	0.8705 (0.0019)	0.0499 (0.0003)
γ_1^*	0.8104 (0.0018)	0.0406 (0.0003)	0.6391 (0.0015)	0.0406 (0.0003)
γ_2	N/A	0.0128 (0.0003)	N/A	0.0128 (0.0003)
γ_2^*	N/A	0.0270 (0.0003)	N/A	0.0270 (0.0003)
ρ^W	-0.2843 (0.0904)		-0.2707 (0.0909)	
ρ^B	0.1817 (0.0678)		0.1982 (0.0643)	
ℓ	36198		37023	

Table 2: Parameter estimates for the two-dimensional two-factor model with seasonality

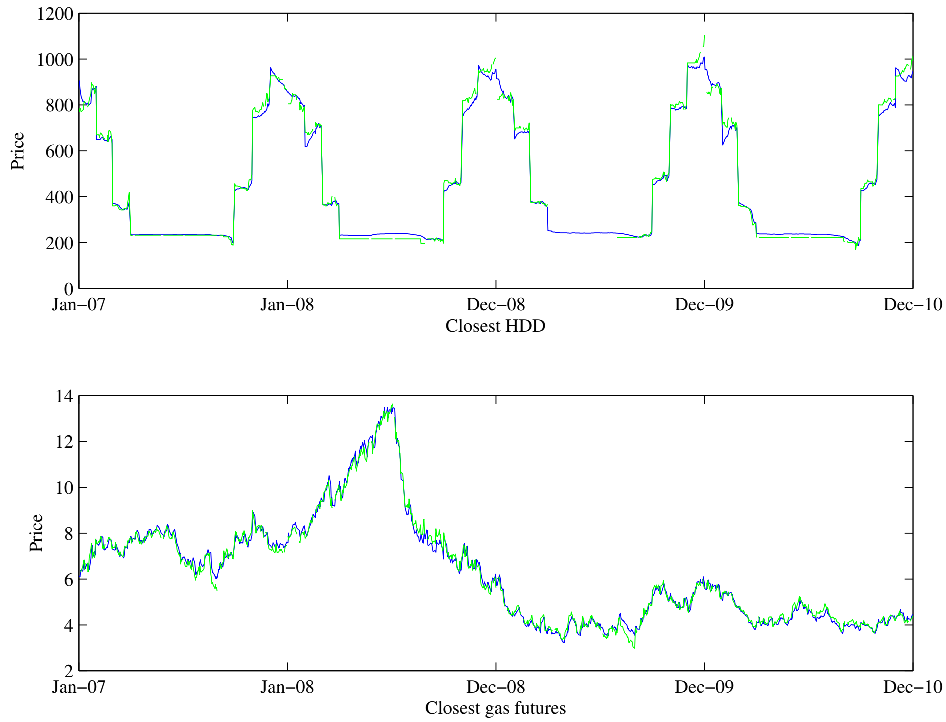

Figure 4: Model prices (blue) and observed prices (green) for the joint estimation of Natural Gas Futures and New York HDDs

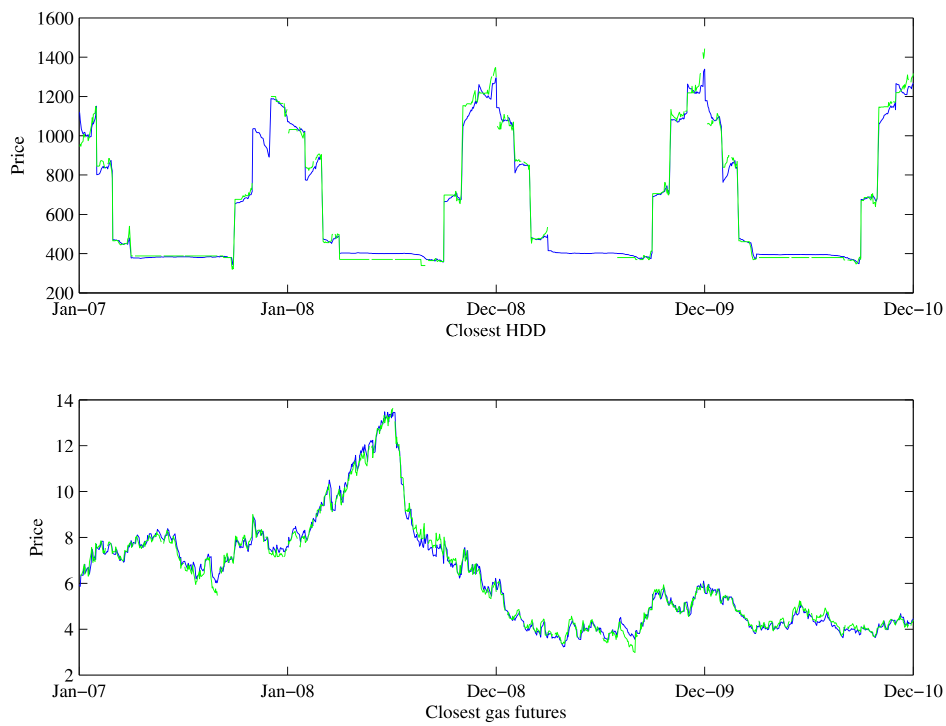

Figure 5: Model prices (blue) and observed prices (green) for the joint estimation of Natural Gas Futures and Chicago HDDs

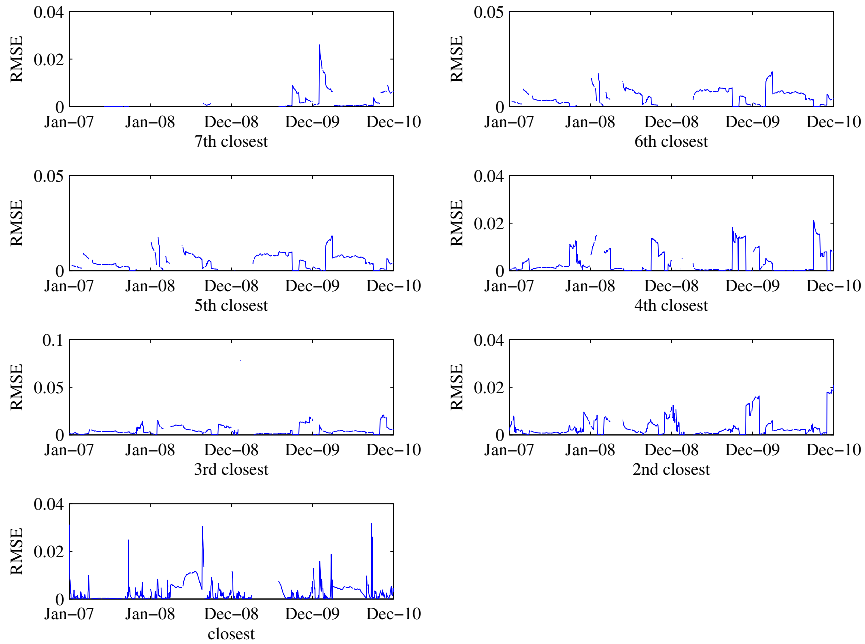

Figure 6: RMSE plot for New York HDD when modelled jointly with Natural Gas

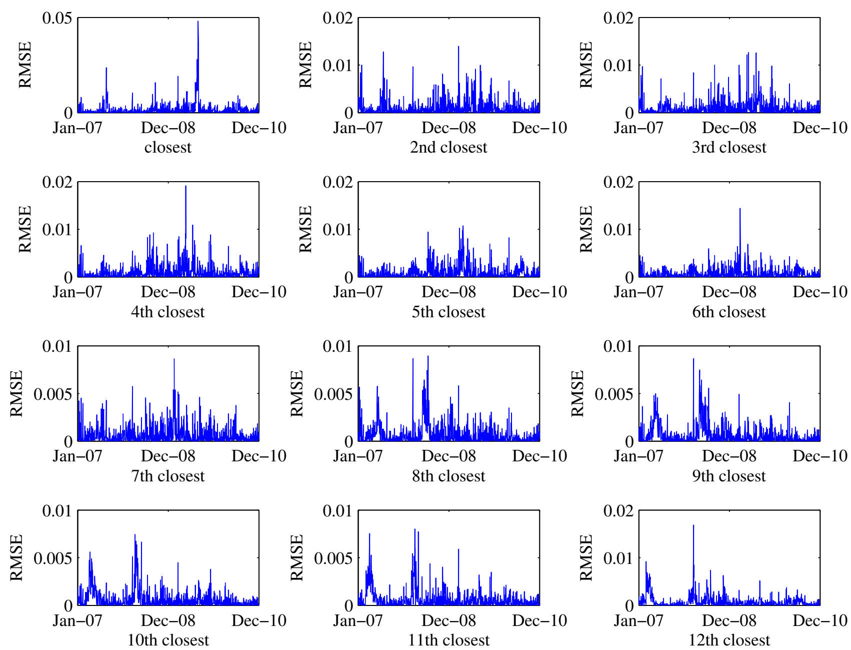

Figure 7: RMSE plot for Natural Gas when modelled jointly with New York HDD

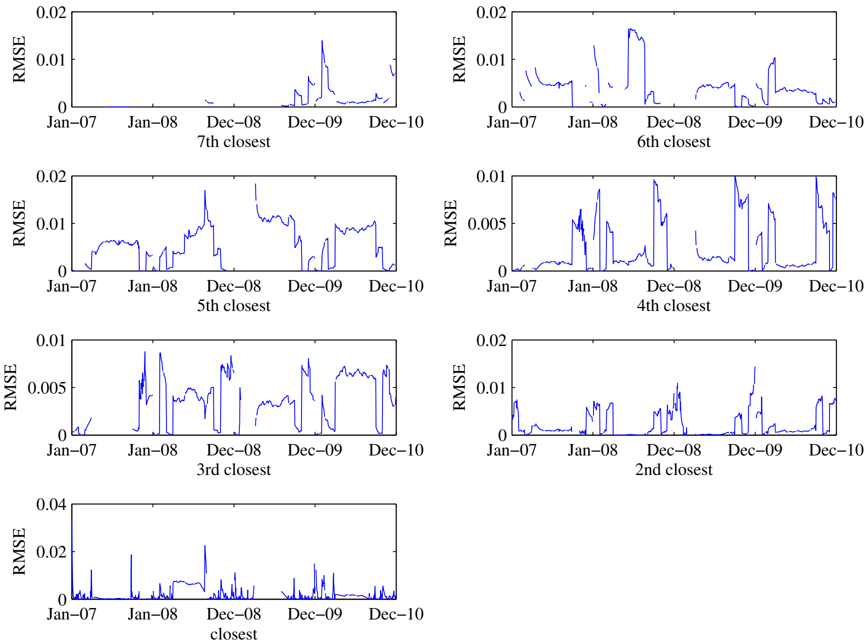

Figure 8: RMSE plot for Chicago HDD when modelled jointly with Natural Gas

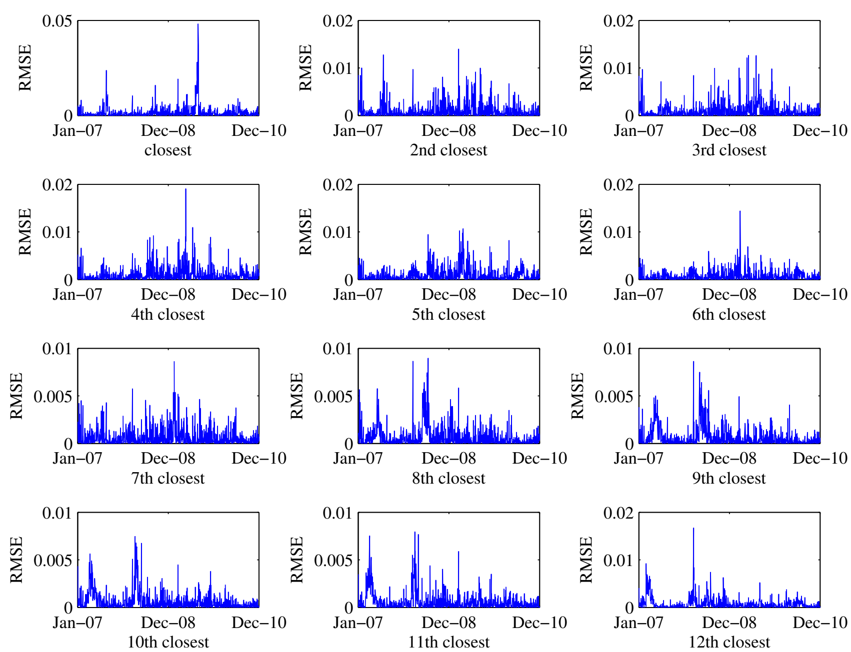

Figure 9: RMSE plot for Natural Gas when modelled jointly with Chicago HDD

4.3 Discussion of Option Prices

To consider the impact of the connection between gas prices and temperature (and thus gas and HDD futures) we compare the quanto option prices with prices obtained under the assumption of independence, and, thus, priced using the model in **?** (see Appendix E). If the two futures were independent, we would get

$$C_t = e^{-r(\tau_2 - t)} \mathbb{E}_{\mathbb{Q}} \left[\max \left(F^E_{\tau_2}(\tau_1, \tau_2) - \overline{K}_E, 0 \right) \right] \times \mathbb{E}_{\mathbb{Q}} \left[\max \left(F^I_{\tau_2}(\tau_1, \tau_2) - \overline{K}_I, 0 \right) \right] ,$$

which can be viewed as the product of the prices of two plain-vanilla call options on the gas and HDD futures respectively. In fact, we have the price C_t given in this case as the product of two Black-76 formulas using the interest rate $r/2$ in the two respective prices.

In Table 3, prices are shown for the quanto option and for the product of the marginal option prices (where we have simply let the correlation between the two futures be zero in the fitted joint model).

$\overline{K}_I \vert \overline{K}_G$	3	4	5	6	7
1100	596 470	451 355	337 270	252 206	188 158
1150	443 325	338 246	254 187	191 142	143 110
1200	401 287	306 217	231 164	173 126	130 97
1250	362 252	277 191	210 145	158 111	118 85
1300	326 222	251 168	190 127	143 97	108 74

Table 3: Option prices for Chicago under the model (top) and under the assumption of no correlation (bottom). $r = 0.02$, τ_1=1-Dec-2011, τ_2=31-Dec-2011, t=31-Dec-2010

From Table 3, it is clear that the correlation between the gas and HDD futures significantly impacts the quanto option price. The fact that the observed correlation increases the quanto option price compared to the product of the two marginal option, indicates that more probability mass lies in the joint exercise region that what the marginal models imply. An alternative to buying the quanto option is to buy a number of gas options. Take for instance the middle quanto option: The price is 231, which has the same cost of buying 169 gas options at price 1.37. In case the gas price is above 5, the holder of the 169 options recieves the gas price less 5 times 169. The holder of the quanto option will recieve the gas price less 5 times the amount of HDDs over 1200. If the total number of HDDs is less that 1369, the holder of the marginal options will receive more, but if the total number of HDDs is above 1369, the holder of the quanto options receives more. We thus see that the quanto option emphasises the more extreme sitations.

5 Conclusion

In this paper we have presented a closed form pricing formula for an energy quanto option under the assumption that the underlying assets are log-normal. Taking advantage of the fact that energy and temperature futures are designed with a delivery period, we show how one can price quanto options using futures contracts as underlying assets. Correspondingly, we adopt an HJM approach, and model the dynamics of the futures contracts directly. We show that our approach encompasses relevant cases, such as geometric Brownian motions and multi-factor spot models. Importantly, our approach enable us to derive hedging strategies and perform hedges with traded assets. We illustrate the use of our pricing model by estimating a two-dimensional two-factor model with seasonality using NYMEX data on natural gas and CME data on temperature HDD futures. We calculate quanto energy option prices and show how correlation between the two asset classes significantly impacts the prices.

A A Comment on Four Correlated Brownian Motions

We have in our two-factor model four Brownian motions W^E, B^E, W^I and B^I. These are correlated as follows: $\rho_E = corr(W_1^E, B_1^E)$, $\rho_I = corr(W_1^I, B_1^I)$, $\rho_W = corr(W_1^E, W_1^I)$ and $\rho_B = corr(B_1^E, B_1^I)$. Moreover, we have cross-correlations given by

$$\begin{aligned} \rho_{I,E}^{W,B} &= corr(W_1^I, B_1^E) \\ \rho_{E,I}^{W,B} &= corr(W_1^E, B_1^I)\,. \end{aligned}$$

We may represent these four correlated Brownian in terms of four independent standard Brownian motions. To this end, introduce the four independent Brownian motions U_E^x, U_E^y, U_I^x and U_I^y. First, we define

$$dW_E = dU_E^x \tag{16}$$

Next, let

$$dB_E = \rho_E dU_E^x + dU_E^y \tag{17}$$

Then we see that $corr(W_1^E, B_1^E) = \rho_E$ as desired. If we define

$$dW_E = \rho_W dU_E^x + (\rho_{I,E}^{W,B} - \rho_E \rho_W) dU_E^y + dU_I^x \tag{18}$$

we find easily that $corr(W_1^E, W_1^I) = \rho_W$ and $corr(B_1^E, W_1^I) = \rho_{I,E}^{W,B}$, as desired. Finally, we define

$$dB_I = \rho_{E,I}^{W,B} dU_E^x + (\rho_B - \rho_E \rho_{E,I}^{W,B}) dU_E^y + c dU_I^x + dU_I^y \tag{19}$$

with

$$c = \rho_I - \rho_W \rho_{E,I}^{W,B} - (\rho_{I,E}^{W,B} - \rho_E \rho_W)(\rho_B - \rho_E \rho_{E,I}^{W,B})\,. \tag{20}$$

With this definition, we find $corr(W_1^E, B_1^I) = \rho_{E,I}^{W,B}$, $corr(B_1^E, B_1^I) = \rho_B$
and $corr(W_1^I, B_1^I) = \rho_I$, as desired.

Note the special case with $\rho_{E,I}^{W,B} = \rho_{I,E}^{W,B} = 0$. Then we have

$$\begin{aligned}
dW_E &= dU_E^x \\
dB_E &= \rho_E dU_E^x + dU_E^y \\
dW_I &= \rho_W dU_E^x - \rho_E \rho_W dU_E^y + dU_I^x \\
dB_I &= \rho_B dU_E^y + (\rho_I + \rho_E \rho_W \rho_B) dU_I^x + dU_I^y
\end{aligned}$$

B The Bivariate Normal Distribution

Assume two random variables X and Y are bivariate normally distributed, i.e.,

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N \left[\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \rho_{xy} \\ \rho_{xy} & \sigma_y^2 \end{pmatrix} \right] \tag{21}$$

where $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2$ and ρ_{xy} denotes the expectations, the variances and the correlation coefficient, respectively. The correlation coefficient ρ_{xy} is defined as

$$\rho_{xy} = \frac{cov(X,Y)}{\sigma_x \sigma_y}. \tag{22}$$

The probability density function (PDF) of the bivariate normal distribution is given by

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} exp\left[-\frac{1}{2\left(1-\rho^2\right)} \left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho_{xy}\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) \right] \right]. \tag{23}$$

The PDF of the bivariate normal distribution could also be written as

$$f(x,y) = f(x) \cdot f(y|x), \tag{24}$$

where $f(x)$ is the marginal density of x, given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_x} exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2 \right], \tag{25}$$

and the density of y conditional on x, $f(y|x)$, is given by

$$f(y|x) = \frac{1}{\sigma_y \sqrt{2\pi} \sqrt{1-\rho_{xy}^2}} exp\left[\frac{1}{2\sigma_y^2\left(1-\rho^2\right)}\left(y-\mu_y-\frac{\rho_{xy}\sigma_y}{\sigma_x}\left(x-\mu_x\right)\right)^2\right]. \tag{26}$$

C Proof of Pricing Formula

In Section 4.1 we showed that the payoff function in (6) could be rewritten in the following way:

$$\begin{aligned}
\hat{p}(F_T^E, F_T^I, \overline{K}_I, \overline{K}_E) &= \max(F_T^I - \overline{K}_I, 0) \cdot \max(F_T^E - \overline{K}_E, 0) \\
&= \left(F_T^E - \overline{K}_E\right) \cdot \left(F_T^I - \overline{K}_I\right) \cdot 1_{\{F_T^E > \overline{K}_E\}} \cdot 1_{\{F_T^I > \overline{K}_I\}} \\
&= F_T^E F_T^I \cdot 1_{\{F_T^E > \overline{K}_E\}} \cdot 1_{\{F_T^I > \overline{K}_I\}} - F_T^E \overline{K}_I \cdot 1_{\{F_T^E > \overline{K}_E\}} \cdot 1_{\{F_T^I > \overline{K}_I\}} \\
&- F_T^I \overline{K}_E \cdot 1_{\{F_T^E > \overline{K}_E\}} \cdot 1_{\{F_T^I > \overline{K}_I\}} + \overline{K}_E \overline{K}_I \cdot 1_{\{F_T^E > \overline{K}_E\}} \cdot 1_{\{F_T^I > \overline{K}_I\}}.
\end{aligned}$$

Now let us calculate the expectation under $\mathbb{Q}$ of the payoff function, i.e., $\mathbb{E}_t^{\mathbb{Q}}\left[\hat{p}(F_T^E, F_T^I, \overline{K}_I, \overline{K}_E)\right]$. We have

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[\hat{p}(F_T^E, F_T^I, \overline{K}_I, \overline{K}_E)\right] &= \mathbb{E}_t^{\mathbb{Q}}\left[\max(F_T^I - \overline{K}_I, 0) \cdot \max(F_T^E - \overline{K}_E, 0)\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[F_T^E F_T^I 1_{\{F_T^E > \overline{K}_E\}} 1_{\{F_T^I > \overline{K}_I\}}\right] - \mathbb{E}_t^{\mathbb{Q}}\left[F_T^E \overline{K}_I 1_{\{F_T^E > \overline{K}_E\}} 1_{\{F_T^I > \overline{K}_I\}}\right] \\
&- \mathbb{E}_t^{\mathbb{Q}}\left[F_T^I \overline{K}_E 1_{\{F_T^E > \overline{K}_E\}} 1_{\{F_T^I > \overline{K}_I\}}\right] + \mathbb{E}_t^{\mathbb{Q}}\left[\overline{K}_E \overline{K}_I 1_{\{F_T^E > \overline{K}_E\}} 1_{\{> \overline{K}_I\}}\right].
\end{aligned} \tag{27}$$

In order to calculate the four different expectation terms we will use the same trick as **?**, namely to rewrite the PDF of the bivariate normal distribution using the identity in (24). Remember that we assume F_T^E and F_T^I to be log-normally distrubuted under $\mathbb{Q}$ (i.e., (X, Y) are bivariate normal):

$$F_T^E = F_t^E e^{\mu_E + X}, \tag{28}$$

$$F_T^I = F_t^I e^{\mu_I + Y}, \tag{29}$$

where σ_X^2 denotes variance of X, σ_Y^2 denotes variance of Y and they are correlated by $\rho_{X,Y}$. Consider the fourth expectation term first,

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[\overline{K}_E\overline{K}_I 1_{\{F_T^E>\overline{K}_E\}}1_{\{F_T^I>\overline{K}_I\}}\right] &= \overline{K}_E\overline{K}_I\mathbb{E}_t^{\mathbb{Q}}\left[1_{\{F_T^E>\overline{K}_E\}}1_{\{F_T^I>\overline{K}_I\}}\right] \\
&= \overline{K}_E\overline{K}_I\mathbb{Q}_t\left(F_T^E>\overline{K}_E \cap\ F_T^I>\overline{K}_I\right) \\
&= \overline{K}_E\overline{K}_I\mathbb{Q}_t\left(F_t^E e^{\mu_E+X}>\overline{K}_E \cap F_t^I e^{\mu_I+Y}>\overline{K}_I\right) \\
&= \overline{K}_E\overline{K}_I\mathbb{Q}_t\left(X>\log\left(\frac{\overline{K}_E}{F_t^E}\right)-\mu_E \cap Y>\log\left(\frac{\overline{K}_I}{F_t^I}\right)-\mu_I\right) \\
&= \overline{K}_E\overline{K}_I\mathbb{Q}_t\left(-X<\log\left(\frac{F_t^E}{\overline{K}_E}\right)+\mu_E \cap -Y<\log\left(\frac{F_t^I}{\overline{K}_I}\right)+\mu_I\right) \\
&= \overline{K}_E\overline{K}_I\cdot M\left(y_1,y_2;\rho_{X,Y}\right),
\end{aligned}$$

where (ϵ_1,ϵ_2) are standard bivariate normal with correlation $\rho_{X,Y}$ and

$$y_1=\frac{\log\left(\frac{F_t^E}{\overline{K}_E}\right)+\mu_E}{\sigma_X} \qquad\qquad y_2=\frac{\log\left(\frac{F_t^I}{\overline{K}_I}\right)+\mu_I}{\sigma_Y}.$$

Next, consider the third expectation term,

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[F_T^I\overline{K}_E 1_{\{F_T^E>\overline{K}_E\}}1_{\{F_T^I>\overline{K}_I\}}\right] &= F_t^I\overline{K}_E e^{\mu_I}\mathbb{E}\left[e^Y 1_{\{F_T^E>\overline{K}_E\}}1_{\{F_T^I>\overline{K}_I\}}\right] \\
&= F_t^I\overline{K}_E e^{\mu_I}\mathbb{E}\left[e^{\sigma_Y\epsilon_2}1_{\{\epsilon_1<y_1\}}1_{\{\epsilon_2<y_2\}}\right] \\
&= F_t^I\overline{K}_E e^{\mu_I}\int_{-\infty}^{y_2}\int_{-\infty}^{y_1} e^{\sigma_Y\epsilon_2}f\left(\epsilon_1,\epsilon_2\right)d\epsilon_1 d\epsilon_2 \\
&= F_t^I\overline{K}_E e^{\mu_I}\int_{-\infty}^{y_2}\int_{-\infty}^{y_1} e^{\sigma_Y\epsilon_2}f\left(\epsilon_2\right)f\left(\epsilon_1|\epsilon_2\right)d\epsilon_1 d\epsilon_2 \\
&= F_t^I\overline{K}_E e^{\mu_I}\int_{-\infty}^{y_2}\int_{-\infty}^{y_1} e^{\sigma_Y\epsilon_2}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\epsilon_2^2\right)\cdot \\
&\qquad \frac{1}{\sqrt{2\pi}\sqrt{1-\rho_{X,Y}^2}}\exp\left[\frac{-1}{2(1-\rho_{X,Y}^2)}\left(\epsilon_1-\rho_{X,Y}\epsilon_2\right)^2\right]d\epsilon_1 d\epsilon_2
\end{aligned} \tag{30}$$

Look at the exponent in the above expression

$$
\begin{aligned}
&\sigma_Y\epsilon_2 - \frac{1}{2}\epsilon_2^2 - \frac{1}{2(1-\rho_{X,Y}^2)}\left(\epsilon_1^2 + \rho_{X,Y}^2\epsilon_2^2 - 2\rho_{X,Y}\epsilon_1\epsilon_2\right) \\
&= -\frac{1}{2(1-\rho_{X,Y}^2)}\left(-2\sigma_Y(1-\rho_{X,Y}^2)\epsilon_2 + (1-\rho_{X,Y}^2)\epsilon_2^2 + \epsilon_1^2 + \rho_{X,Y}^2\epsilon_2^2 - 2\rho_{X,Y}\epsilon_1\epsilon_2\right) \\
&= -\frac{1}{2(1-\rho_{X,Y}^2)}\left(\epsilon_1^2 - 2\sigma_Y(1-\rho_{X,Y}^2)\epsilon_2 + \epsilon_2^2 - 2\rho_{X,Y}\epsilon_1\epsilon_2\right) \\
&= -\frac{1}{2(1-\rho_{X,Y}^2)}\left(w^2 + z^2 - 2\rho_{X,Y}zw - (1-\rho_{X,Y}^2)\sigma_Y^2\right) \\
&= -\frac{1}{2(1-\rho_{X,Y}^2)}\left(w^2 + z^2 - 2\rho_{X,Y}zw\right) + \frac{\sigma_Y^2}{2}
\end{aligned}
$$

using the substitution $w = -\epsilon_1 + \rho_{\epsilon_1,\epsilon_2}\sigma_Y$ and $z = -\epsilon_2 + \sigma_Y$, (30) can be written as

$$
\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}\left[F_T^I\overline{K}_E 1_{\{F_T^E > \overline{K}_E\}} 1_{\{F_T^I > \overline{K}_I\}}\right] = \\
&\quad F_t^I\overline{K}_E e^{\mu_I + \frac{\sigma_Y^2}{2}} \int_{-\infty}^{y_2^*}\int_{-\infty}^{y_1^*} \frac{1}{2\pi\sqrt{1-\rho_{X,Y}^2}} \exp\left[-\frac{1}{2(1-\rho_{X,Y}^2)}\left(w^2 + z^2 - 2\rho_{X,Y}zw\right)\right] dwdz \\
&\quad = F_t^I\overline{K}_E e^{\mu_I + \frac{\sigma_Y^2}{2}} M\left(y_1^*, y_2^*; \rho_{X,Y}\right)
\end{aligned}
$$

where

$$
y_1^* = y_1 + \rho_{X,Y}\sigma_Y \qquad\qquad y_2^* = y_2 + \sigma_Y.
$$

The second expectation term can be calculated in the same way as we calculated the third term. The only difference is that we now use the substitution $\bar{w} = -\epsilon_1 + \sigma_X$ and $\bar{z} = -\epsilon_2 + \rho_{X,Y}\sigma_X$, so we can write

$$
\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}\left[F_T^E\overline{K}_I 1_{\{F_T^E > \overline{K}_E\}} 1_{\{F_T^I > \overline{K}_I\}}\right] = \\
&\quad F_t^E\overline{K}_I e^{\mu_E + \frac{\sigma_X^2}{2}} \int_{-\infty}^{y_2^{**}}\int_{-\infty}^{y_1^{**}} \frac{1}{2\pi\sqrt{1-\rho_{X,Y}^2}} \exp\left[-\frac{1}{2(1-\rho_{X,Y}^2)}\left(w^2 + z^2 - 2\rho_{X,Y}zw\right)\right] dwdz \\
&\quad = F_t^E\overline{K}_I e^{\mu_E + \frac{\sigma_X^2}{2}} M\left(y_1^{**}, y_2^{**}; \rho_{X,Y}\right)
\end{aligned}
$$

where

$$
y_1^{**} = y_1 + \sigma_X \qquad\qquad y_2^{**} = y_2 + \rho_{X,Y}\sigma_X.
$$

Finally, consider the first expectation term in (27),

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[F_T^E F_T^I 1_{\{F_T^E>\overline{K}_E\}}1_{\{F_T^I>\overline{K}_I\}}\right] &= F_t^E F_t^I e^{\mu_E+\mu_I}\mathbb{E}_t^{\mathbb{Q}}\left[e^{X+Y}1_{\{F_T^E>\overline{K}_E\}}1_{\{F_T^I>\overline{K}_I\}}\right]\\ &= F_t^E F_t^I e^{\mu_E+\mu_I}\mathbb{E}_t^{\mathbb{Q}}\left[e^{\sigma_X\epsilon_1+\sigma_Y\epsilon_2}1_{\{\epsilon_1<y_1\}}1_{\{\epsilon_2<y_2\}}\right]\\ &= F_t^E F_t^I e^{\mu_E+\mu_I}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2} e^{\sigma_X\epsilon_1+\sigma_Y\epsilon_2}f(\epsilon_1,\epsilon_2)d\epsilon_2 d\epsilon_1\end{aligned} \tag{31}$$

Using the same trick as before with the substitution $u=-\epsilon_1+\rho_{X,Y}\sigma_Y+\sigma_X$ and $v=-\epsilon_2+\rho_{X,Y}\sigma_X+\sigma_Y$, expression (31) can be written

$$u^2+v^2-2\rho_{X,Y}uv-(1-\rho_{x,Y}^2)\left(\sigma_X^2+\sigma_Y^2+2\rho_{X,Y}\sigma_X\sigma_Y\right)=\epsilon_1^2+\epsilon_2^2-2\rho_{X,Y}\epsilon_2\epsilon_1-2(1-\rho_{X,Y}^2)\sigma_X\epsilon_1-2(1-\rho_{X,Y}^2)\sigma_Y\epsilon_2$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[F_T^E F_T^I 1_{\{F_T^E>\overline{K}_E\}}1_{\{F_T^I>\overline{K}_I\}}\right] = F_t^E F_t^I e^{\mu_E+\mu_I+\frac{1}{2}(\sigma_X^2+\sigma_Y^2+2\rho_{X,Y}\sigma_X\sigma_Y)}M(y_1^{***},y_2^{***};\rho_{X,Y}) \tag{32}$$

where

$$y_1^{***}=y_1+\rho_{X,Y}\sigma_Y+\sigma_X \qquad\qquad y_2^{***}=y_2+\rho_{X,Y}\sigma_X+\sigma_Y.$$

Thus the expectation of the payoff function is

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[\hat{p}(F_T^E,F_T^I,\overline{K}_I,\overline{K}_E)\right] &= F_t^E F_t^I e^{\mu_E+\mu_I+\frac{1}{2}(\sigma_X^2+\sigma_Y^2+2\rho_{X,Y}\sigma_X\sigma_Y)}M(y_1^{***},y_2^{***};\rho_{X,Y})\\ &- F_t^E\overline{K}_I e^{\mu_E+\frac{\sigma_X^2}{2}}M\left(y_1^{**},y_2^{**};\rho_{X,Y}\right)\\ &- F_t^I\overline{K}_E e^{\mu_I+\frac{\sigma_Y^2}{2}}M\left(y_1^{*},y_2^{*};\rho_{X,Y}\right)\\ &+ \overline{K}_E\overline{K}_I\cdot M\left(y_1,y_2;\rho_{X,Y}\right),\end{aligned}$$

Discounting the expected payoff under gives us the price of the option. End of proof.

D Closed Form Solutions for σ and ρ in the two-dimensional Schwartz-Smith Model with Seasonality

$$
\begin{aligned}
\sigma_X^2 &= \int_t^T \left(\sigma_E^2 + \left(\nu_E e^{-\kappa^E(\tau-s)} \right)^2 + 2\rho_E \sigma_E \left(\nu_E e^{-\kappa^E(\tau-s)} \right) \right) ds \\
&= \sigma_E^2 (T-t) + \nu_E \int_t^T e^{-2\kappa^E(\tau-s)} ds + 2\rho_E \sigma_E \nu_E \int_t^T e^{-\kappa^E(\tau-s)} ds \\
&= \sigma_E^2 (T-t) + \frac{\nu_E}{2\kappa^E} e^{-2\kappa^E \tau} \left(e^{2\kappa^E T} - e^{2\kappa^E t} \right) + 2 \frac{\rho_E \sigma_E \nu_E}{\kappa^E} e^{-\kappa^E \tau} \left(e^{\kappa^E T} - e^{\kappa^E t} \right) \\
cov(X,Y) &= \rho_W \int_t^T \sigma_E \sigma_I ds + \rho_B \int_t^T \left(\nu_E e^{-\kappa^E(\tau-s)} \right) \left(\nu_I e^{-\kappa^I(\tau-s)} \right) ds \\
&= \rho_W \sigma_E \sigma_I (T-t) + \rho_B \nu_E \nu_I e^{-(\kappa^E+\kappa_I)\tau} \int_t^T e^{(\kappa^E+\kappa_I)s} ds \\
&= \rho_W \sigma_E \sigma_I (T-t) + \frac{\rho_B \nu_E \nu_I}{\kappa^E + \kappa^I} e^{-(\kappa^E+\kappa_I)\tau} \left(e^{(\kappa^E+\kappa^I)T} - e^{(\kappa^E+\kappa^I)t} \right) \\
\rho_{X,Y} &= \frac{cov(X,Y)}{\sigma_X \sigma_Y}
\end{aligned}
$$

When $T = \tau$, this simplifies to

$$
\begin{aligned}
\sigma_X &= \sigma_E^2 (\tau - t) + \frac{\nu_E}{2\kappa^E} \left(1 - e^{-2\kappa^E(\tau-t)} \right) + 2 \frac{\rho_E \sigma_E \nu_E}{\kappa^E} \left(1 - e^{\kappa^E(\tau-t)} \right) \\
\rho_{X,Y} &= \frac{\rho_W \sigma_E \sigma_I (\tau-t) + \frac{\rho_B \nu_E \nu_I}{\kappa^E+\kappa^I} \left(1 - e^{-(\kappa^E+\kappa_I)(\tau-t)} \right)}{\sigma_X \sigma_Y}
\end{aligned}
$$

E One-dimensional Option Prices

As before, assume that the dynamics of a gas futures contract is given by:

$$F_T^E(\tau_1, \tau_2) = F_t^E(\tau_1, \tau_2) \exp(\mu_E + X).$$

Consider now a call option written on gas futures only. The price c_t of this option is then given by the Black-76 formula, i.e.

$$c_t = e^{-r(T-t)} \left[F N(d_1) - K N(d_2) \right],$$

where

$$d_1 = \frac{\ln \frac{F_t^E}{K_E} - \mu_E}{\sigma_X} \qquad\qquad d_2 = \frac{\ln \frac{F_t^E}{K_E} + \mu_E}{\sigma_X}.$$

The same formula of course applies to an option written only on temperature futures.

F Estimation Using Kalman Filter Techniques

Given a set of observed futures prices, it is possible to estimate the parameters using Kalman filter techniques. Let

$$Y_n = \left(f^I_{t_n}\left(T^1_n\right), \ldots, f^I_{t_n}\left(T^{M^I_n}_n\right), f^E_{t_n}\left(T^1_n\right), \ldots, f^E_{t_n}\left(T^{M^E_n}_n\right) \right)'$$

denote the set of log-futures prices observed at time t_n with maturities $T^1_n, \ldots, T^{M^I_n}_n$ for the temperature contracts and maturitues $T^1_n, \ldots, T^{M^E_n}_n$ for the gas contracts. The measurement equation relates the observations to the unobserved state vector $U_n = (X_{t_n}, Z_{t_n})'$ by

$$Y_n = d_n + C_n U_n + \epsilon_n$$

where the ϵ's are measurement errors assumed i.i.d. normal with zero mean and covariance matrix H_n. In the present framework we have

$$d_n = \begin{pmatrix} \Lambda^I\left(T^1_n\right) + A^I\left(T^1_n - t_n\right) \\ \vdots \\ \Lambda^I\left(T^{M^T_n}_n\right) + A^I\left(T^{M^I_n}_n - t_n\right) \\ \Lambda^E\left(T^1_n\right) + A^E\left(T^1_n - t_n\right) \\ \vdots \\ \Lambda^E\left(T^{M^G_n}_n\right) + A^E\left(T^{M^E_n}_n - t_n\right) \end{pmatrix}, C_n = \begin{pmatrix} 1 & e^{-\kappa^I\left(T^1_n - t_n\right)} \\ \vdots & \vdots \\ 1 & e^{-\kappa^I\left(T^{M^I_n}_n - t_n\right)} \\ 1 & e^{-\kappa^E\left(T^1_n - t_n\right)} \\ \vdots & \vdots \\ 1 & e^{-\kappa^E\left(T^{M^E_n}_n - t_n\right)} \end{pmatrix}$$

and $H_n = \begin{pmatrix} \sigma^2_{\epsilon,I} I_{M^I_n} & 0 \\ 0 & \sigma^2_{\epsilon,E} I_{M^E_n} \end{pmatrix}$

The state-vector evolves according to

$$U_n = c + TU_n + \eta_n$$

where η_n are i.i.d. normal with zero-mean vector and covariance matrix Q and where

$$c = \begin{pmatrix} \mu^I - \frac{1}{2}\left(\sigma^I\right)^2 \\ 0 \\ \mu^E - \frac{1}{2}\left(\sigma^E\right)^2 \\ 0 \end{pmatrix} \Delta_{n+1}, T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\kappa^I \Delta_{n+1}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-\kappa^E \Delta_{n+1}} \end{pmatrix}$$

$$Q = \begin{pmatrix} \left(\sigma^I\right)^2 \Delta_{n+1} & 0 & \rho^S \sigma^I \sigma^E \Delta_{n+1} & 0 \\ 0 & \frac{\left(\nu^I\right)^2\left(1-e^{-2\kappa^I \Delta_{n+1}}\right)}{2\kappa^I} & 0 & \rho^L \frac{\nu^I \nu^E \left(1-e^{-\left(\kappa^I+\kappa^E\right)\Delta_{n+1}}\right)}{\left(\kappa^I+\kappa^E\right)} \\ \rho^S \sigma^I \sigma^E \Delta_{n+1} & 0 & \left(\sigma^E\right)^2 \Delta_{n+1} & 0 \\ 0 & \rho^L \frac{\nu^I \nu^E \left(1-e^{-\left(\kappa^I+\kappa^E\right)\Delta_{n+1}}\right)}{\left(\kappa^I+\kappa^E\right)} & 0 & \frac{\left(\nu^E\right)^2\left(1-e^{-2\kappa^E \Delta_{n+1}}\right)}{2\kappa^E} \end{pmatrix}$$