PRICING AND HEDGING OPTIONS IN ENERGY MARKETS BY BLACK-76

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Abstract. We prove that the price of options on forwards in commodity markets converge uniformly to the Black-76 formula when the short-term variations of the logarithmic spot price is a stationary Ornstein-Uhlenbeck process and the long-term variations are following a drifted Brownian motion. The convergence rate is exponential in the speed of mean-reversion and time to delivery of the underlying forward from the exercise time of the option. This can be applied to energy markets like electricity and gas to argue for the use of Black-76 in pricing of options, although the spot prices may show large spikes. Furthermore, we prove that the quadratic hedging strategy converges in a similar fashion to the delta-hedge in the Black-76 model. Our results are illustrated with a numerical example of relevance to energy markets.

1. Introduction

The typical stochastic models for spot and forward prices in oil, gas and electricity markets separate the time evolution into long-term and short-term factors. The long-term effects include inflation and depletion of reserves (in case of non-renewable commodities like oil and gas), and is typically thought of as being non-stationary. On the other hand, the prices are shocked by short-term effects like outages of power plants or changes in demand from temperature variations. These effects are modelled by stationary, mean-reverting processes. The classical model of Gibson and Schwartz [12] defines the logarithmic oil spot prices as a drifted Brownian motion and an Ornstein-Uhlenbeck process. This two-factor model, consisting of a non-stationary and a stationary part, has later been applied to gas and electricity markets (see e.g. Lucia and Schwartz [13]), in particular to forward pricing.

In energy markets like gas and power, options traded on exchanges are typically written on futures contracts delivering the underlying energy over a specified period. For example, at the Nordic power market NordPool call and put options are traded based on futures contract with financial delivery of electricity over given months. One may suspect that the short-term factor of the forward price evolution inherited from the spot will be insignificant in the option price. Due to the delivery period, short-term shocks in the spot may vanish in the futures dynamics due to smoothing by the delivery period. This means that one is left with the non-stationary part, which leads to the claim that the option price can be approximated well by the Black-76 formula. In this paper we show that this is indeed true in many practically relevant situations. In fact, we prove a uniform exponential convergence of the "true" option price towards the one given by Black-76 in terms of the speed of mean reversion of the short-term stationary factor and the time left to delivery of the underlying futures from the exercise time of the option. A "folklore" in the NordPool market says that one can do well with Black-76. We show that this is indeed the case, justified by theoretical results and numerical examples.

In Lucia and Schwartz [13], the forward price dynamics for contracts delivering electricity over a specified period is defined as the average of forwards with fixed delivery time. As the spot model is defined as an exponential process, there exists no analytic formula for the forward price delivering over a period for models of interest. In this paper we view this differently, and think of the forward price with delivery period as a contract with "fixed-delivery" given by the mid-point

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of the delivery period. In this way we can make a reasonable approximation of the forward price dynamics in electricity and gas, where one does not have the classical convergence of forward price to spot when time to start of delivery goes to zero (see Benth et al. [5] for more details). Obviously, for other commodities (like oil), where the forward delivers at a given time we do not need to use such an approach.

A typical characteristic of gas and electricity markets are sudden large price deviations, frequently referred to as spikes. For example, the German power market EEX shows a significant amount of negative price spikes, mainly due to wind power generation. More usual are the positive spikes, which for example can be seen in the NordPool market during winter season. Also in gas markets one sees large price fluctuations occuring due to for example cold weather (see for example Geman [11] for a discussion). These big price fluctuations call for models based on non-Gaussian stochastic drivers, and the application of Lévy processes, possibly time-inhomogeneous, seems natural (see Benth et al. [5]). In this paper we model the short-term dynamics by an Ornstein-Uhlenbeck process driven by a Lévy process. In this way we can include modelling of spikes, or large variations, in the price dynamics.

The implication of a spot price driven by Lévy innovations is that the forward price dynamics become more involved. Also, we are put in an incomplete market setting which makes pricing and hedging of the option a delicate problem. As we choose to introduce a pricing measure Q based on the Girsanov and Esscher transforms (see Benth et al. [5]), we have already pinned down a risk-neutral probability for the forward prices (namely the one we choose when deriving the forward from the spot). We can then derive the call option price based on a conditional expectation of the payout from the call. On the other hand, there exists no hedging strategy perfectly replicating the option.

There exists many approaches to hedging in incomplete markets, where one soughts to find a strategy in the underlying which minimizes the risk exposed in a short position of the option (see Cont and Tankov [7]). We focus here on the quadratic hedging strategy, which minimizes the L^2-distance between the payout from the option and the hedging portfolio. We refer to Cont and Tankov [7] for more on this strategy in incomplete markets where the underlying asset price is defined as an exponential Lévy process. We are able to determine the quadratic hedging strategy for our market model, and express this in terms of the option price and its sensitivity to the underlying (the delta). As it turns out, we are able to show that the quadratic hedge converges uniformly to the simple delta-hedging strategy, and moreover, we determine the rate of convergence to be the same as for the price, namely exponential in time to delivery and mean-reversion speed.

Our findings are presented as follows. In the next Section we introduce our spot price model and derive the forward price dynamics. Section 3 deals with the convergence of option prices towards the Black-76. This Section also presents a Fourier-based pricing formula as well as a numerical illustration. Next, in Section 4, we derive the quadratic hedging strategy for call options on forwards, and prove that this converges exponentially to the delta hedge of Black-76.

2. The spot and forward price dynamics

Fix a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, and suppose that the energy spot price follows a two factor model defined as

$$S(t) = \Lambda(t)\exp(X(t) + Y(t))\,. \tag{2.1}$$

Here, the non-stationary factor X is a drifted Brownian motion

$$dX(t) = \mu\,dt + \sigma\,dB(t)\,, \tag{2.2}$$

with B being a Brownian motion and $\mu, \sigma > 0$ constants. The stationary factor Y is given by the Ornstein-Uhlenbeck dynamics

$$dY(t) = -\beta Y(t)\,dt + dL(t)\,, \tag{2.3}$$

where L is a pure jump Lévy process with Lévy-Khintchine decomposition

$$L(t) = \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(dt, dz)\,,$$

and $\beta > 0$ a constant. The deterministic seasonality function $\Lambda(t) : \mathbb{R}_+ \to \mathbb{R}_+$ is supposed to be continuous.

The exponential two-factor dynamics (2.1) is a generalization of the spot price model proposed by Gibson and Schwartz [12] (see also Schwartz and Smith [15]). They assumed L to be a Brownian motion correlated with B, and applied the model to a study of oil spot and forward prices. Later, Lucia and Schwartz [13] suggested such a two factor model for electricity spot and forward prices, again using L as a Brownian motion. They studied empirically NordPool data. The two-factor model takes into account mean reversion of the commodity price as well as uncertainty in the equilibrium level to which the prices revert. The non-stationary long time factor models the equilibrium price level, and reflects expectations on for example improving technologies for the production of the commodity, inflation or political and regulatory effects, and depletion of non-renewable resources like gas and coal. The mean reverting short term factor describes changes in demand and supply resulting for example from variations in the weather conditions and sudden outages of power plants. They are tempered by the ability of market participicants to respond to the changing market conditions and are therefore reverting back to their mean level. Lucia and Schwartz [13] provide evidence that one finds seasonal regular patterns in the electricity spot prices, accounted for in the model by the function Λ.

To make our analysis slightly simpler, we shall assume that L and B are independent. Moreover, as already stated, we let L be a pure-jump Lévy process and denote its Lévy measure by $\ell(dz)$. The motivation behind assuming a Lévy process rather than a Brownian motion driving the stationary part comes from power markets, where the spot prices are known to have spikes. Such spikes are typically of short duration, and can be reasonably well modelled by a jump (in the Lévy process) followed by fast mean reversion (coming from a large β). Also in gas markets one expects big short term variations, where a Lévy process seems more natural to span the undertainty than a Brownian motion driven Ornstein-Uhlenbeck process. We refer to Benth et al. [5] for more discussions motivating the use of jump processes in energy markets.

We assume that the Lévy process has finite exponential moments, that is,

$$\int_1^\infty \mathrm{e}^{cz}\,\ell(dz) + \int_{-\infty}^{-1} \mathrm{e}^{-cz}\,\ell(dz) < \infty\,, \tag{2.4}$$

for a positive constant c. As we shall see, we need to have the constant $c \geq 3$ in order to prove our results. Hence, we suppose that this is true from now on. Finally, we denote by ϕ the logarithmic moment generating function of $L(1)$, defined as

$$\phi(\theta) = \ln \mathbb{E}[\exp(\theta L(1))]\,, \tag{2.5}$$

which exists for $|\theta| \leq 3$.

Since our attention is on pricing call option written on forward contracts, we need to relate the forward price dynamics to the spot model. The standard definition of the forward price $f(t,T)$ at time $t \geq 0$ of a contract delivering the underlying energy at time $T \geq t$ is

$$f(t,T) = \mathbb{E}_Q[S(T) \,|\, \mathcal{F}_t] \tag{2.6}$$

for some pricing measure Q being equivalent to P. We implicitly assume here that $S(T)$ is integrable with respect to the pricing measure Q. In electricity, say, the spot is not storable, and *any* equivalent measure Q can be used as a pricing measure (see Benth et al. [5]). Gas can be stored and traded in a spot market, but transportation and storage costs will be incurred. The same is the case of oil. In addition, one talks about the convenience yield for these commodities. Collected together, one may view the storage costs, transportation and convenience yield as a result of a measure change, or, vice versa, that a measure change from P to Q is a modelling of these three components. Thus, also in the gas and oil situation, it is convenient to define a rich class of equivalent probability measures which can flexibly model the drift imposed by storage, transportation and convenience yield. The standard class of probabilities is provided by the Esscher transform, which coincides with the Girsanov transform for the Brownian motion case. Using a constant Esscher transform (see Benth et al. [5]), the effect on the stationary factor is an additional drift coefficient adding on the μ, and for the Lévy process the effect will be an exponential tilting of the Lévy measure, but preserving the Lévy property. Hence, in order to keep notation at a minimum, we suppose that our spot model is already stated under a pricing measure Q (or, we can just re-interpret the meaning of the coefficients in the spot model).

The next proposition states the forward price explicitly in terms of the logarithmic moment generating function of $L(1)$.

Proposition 2.1. *The forward price $f(t,T)$ at $t \geq 0$ with delivery at $T \geq t$ is*

$$f(t,T) = h(t,T)\exp\left(X(t) + e^{-\beta(T-t)}Y(t)\right)$$

with

$$h(t,T) = \Lambda(T)\exp\left(\mu(T-t) + \frac{1}{2}\sigma^2(T-t) + \int_t^T \phi(e^{-\beta(T-s)})\,ds\right).$$

Proof. First, notice that

$$X(T) = X(t) + \mu(T-t) + \sigma(B(T) - B(t)),$$

and

$$Y(T) = \mathrm{e}^{-\beta(T-t)}Y(t) + \int_t^T \mathrm{e}^{-\beta(T-s)}\,dL(s)$$

by a straightforward use of the Itô formula for jump processes. But then, by the $\mathcal{F}_t$-adaptedness of $X(t)$ and $Y(t)$, the independent increment property of Lévy processes and the independence between B and L, we find

$$\begin{aligned} f(t,T) &= \Lambda(T)\mathbb{E}[\exp(X(T) + Y(T))\,|\,\mathcal{F}_t] \\ &= \Lambda(T)\exp\left(\mu(T-t) + X(t) + \mathrm{e}^{-\beta(T-t)}Y(t)\right)\mathbb{E}\left[\exp(\sigma(B(T) - B(t))\right] \\ &\quad\times \mathbb{E}\left[\exp(\int_t^T \mathrm{e}^{-\beta(T-s)}\,dL(s))\right] \\ &= h(t,T)\exp\left(X(t) + \mathrm{e}^{-\beta(T-t)}Y(t)\right). \end{aligned}$$

This proves the result. □

We can find the dynamics of the forward price, which shows that it is indeed a geometric jump-diffusion model:

Proposition 2.2. *The dynamics of the process $t \mapsto f(t,T)$ for $t \leq T$ is*

$$\frac{df(t,T)}{f(t-,T)} = \sigma\,dB(t) + \int_{\mathbb{R}}\left\{\exp\left(ze^{-\beta(T-t)}\right) - 1\right\}\widetilde{N}(dz,dt),$$

where $\widetilde{N}(dt,dz)$ is the compensated Poisson random measure of L and $f(t-,T)$ denotes the left-limit of $f(t,T)$.

Proof. Observe that f has finite expectation using (2.4) and that by definition, $t \mapsto f(t,T)$ is a martingale. This information simplifies considerably the application of Itô's Formula for jump processes, which shows the result. □

We remark that there are several papers modelling forward prices in energy directly rather than as a derivative of the spot price dynamics. A direct modelling of forward prices, following the so-called Heath-Jarrow-Morton approach from interest rate theory, has been extensively discussed in Benth et al. [5], as well as Benth and Koekebakker [4]. One natural class of such models may in fact be the dynamical model stated in Prop. 2.2.

3. Pricing call options on forwards

With the forward price at hand we go on and analyse the price of options on forwards. We focus our attention on European call options, and remark that put options can be priced via the call-put parity (see Benth et al. [5]).

To this end, we let $\tau \leq T$ be the exercise time of the call option, with a strike price $K > 0$. To simplify the exposition slightly, we assume that the risk-free interest rate is equal to zero, that is, $r = 0$. The no-arbitrage price of a call option at time $t \leq \tau$ written on a forward contract with price dynamics given as in Prop. 2.1, or equivalently Prop. 2.2, is defined by

$$C(t, \tau, T) = \mathbb{E}\left[\max(f(\tau, T) - K, 0) \,|\, \mathcal{F}_t\right] . \tag{3.1}$$

By Prop. 2.2, we see that the forward price is Markovian, and hence we find that the price of the call can be expressed as $C(t, \tau, T, f(t, T))$, with $C(t, \tau, T, x)$ given by

$$C(t, \tau, T, x) = \mathbb{E}\left[\max(f(\tau, T) - K, 0) \,|\, f(t, T) = x\right] . \tag{3.2}$$

Our aim now is to analyse this price in relation to the Black-76 formula. For the convenience of the reader, we have stated this famous formula for the price of a call option written on a forward with a geometric Brownian motion dynamics (see Black [6]).

Proposition 3.1. *Suppose the forward price dynamics is a geometric Brownian motion*

$$\frac{df(t, T)}{f(t, T)} = \sigma\, dB(t) .$$

Then the price at time t of a call option with strike K and exercise time $t \leq \tau \leq T$, is given by $C_{B76}(t, f(t, T))$ with

$$C_{B76}(t, \tau, T, x) = x\Phi(d_1(x)) - K\Phi(d_2(x))$$

for Φ being the cumulative standard normal distribution function, and

$$d_1(x) = d_2 + \sigma\sqrt{\tau - t}$$
$$d_2(x) = \frac{\ln\left(\frac{x}{K}\right) - \frac{1}{2}\sigma^2(\tau - t)}{\sigma\sqrt{\tau - t}} .$$

Proof. See Black [6]. □

We want to show that $C(t, \tau, T, x)$ is converging to $C_{\mathrm{B76}}(t, \tau, T, x)$ as the delivery time T of the underlying forward goes to infinity. Moreover, we want to have the rate of convergence measured in terms of the speed of mean reversion β of the spike component.

The price $C(t, \tau, T, x)$ can be represented as follows:

Proposition 3.2. *The price of a call option on the forward given in Prop. 2.1 is*

$$\begin{aligned} &C(t, \tau, T, x) \\ &\quad = x\mathbb{E}\left[\exp\left(\int_t^\tau e^{-\beta(T-s)}\, dL(s) - \int_t^\tau \phi(e^{-\beta(T-s)})\, ds\right) \Phi\left(d_1\left(x, \int_t^\tau e^{-\beta(T-s)}\, dL(s)\right)\right)\right] \\ &\qquad - K\mathbb{E}\left[\Phi\left(d_2\left(x, \int_t^\tau e^{-\beta(T-s)}\, dL(s)\right)\right)\right] \end{aligned}$$

where $\phi(x)$ is the logarithmic moment generating function of $L(1)$ and

$$d_1(x, v) = d_2(x, v) + \sigma\sqrt{\tau - t}$$
$$d_2(x, v) = \frac{\ln\left(\frac{x}{K}\right) + v - \int_t^\tau \phi(e^{-\beta(T-s)})\, ds - \frac{1}{2}\sigma^2(\tau - t)}{\sigma\sqrt{\tau - t}} .$$

Proof. First, from Prop. 2.1, we have

$$\begin{aligned} f(\tau, T) &= h(\tau, T) \exp\left(X(\tau) + \mathrm{e}^{-\beta(T-\tau)} Y(\tau)\right) \\ &= f(t, T)\frac{h(\tau, T)}{h(t, T)} \exp\left(X(\tau) - X(t) + \mathrm{e}^{-\beta(T-\tau)} Y(\tau) - \mathrm{e}^{-\beta(T-t)} Y(t)\right) . \end{aligned}$$

But,

$$\mathrm{e}^{-\beta(T-\tau)}Y(\tau) = Y(t)\mathrm{e}^{-\beta(T-t)} + \int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)$$

and

$$X(\tau) - X(t) = \mu(\tau - t) + \sigma(B(\tau) - B(t))\,.$$

Furthermore,

$$\frac{h(\tau,T)}{h(t,T)}\mathrm{e}^{\mu(\tau-t)} = \exp\left(-\frac{1}{2}\sigma^2(\tau-t) - \int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\,ds\right)\,.$$

Hence,

$$\begin{aligned} f(\tau,T) = f(t,T)\exp&\left(\sigma(B(\tau)-B(t)) - \frac{1}{2}\sigma^2(\tau-t)\right)\\ &\times\exp\left(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s) - \int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\,ds\right)\,. \end{aligned}$$

Denote by $Z(x)$ the random variable

$$Z(x) = x\exp\left(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s) - \int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\,ds\right)\,,$$

and since L is independent of B, $Z(x)$ is independent of $B(\tau)-B(t)$. Conditioning on Z, yields

$$\begin{aligned} C(t,\tau,T,x) &= \mathbb{E}\left[\max(f(\tau,T)-K,0)\,|\,x = f(t,T)\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[\max\left(Z(x)\exp\left(\sigma(B(\tau)-B(t)) - \frac{1}{2}\sigma^2(\tau-t)\right) - K, 0\right)\,|\,Z(x)\right]\right]\,. \end{aligned}$$

The inner expectation can be computed by the Black-76 formula in Prop. 3.1, with $Z(x)$ playing the role of x. Hence, the result follows. □

The expression for the price in the proposition above can now be used to show the convergence to the Black-76 formula. We prove this by a sequence of Lemmas. But first, let us introduce the log-moment generating function

$$\phi_\beta(\theta) := \ln\mathbb{E}\left[\exp\left(\int_t^\tau \theta\mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right]\,. \tag{3.3}$$

From Benth et al. [5] we have

$$\phi_\beta(\theta) = \int_t^\tau \phi\left(\theta\mathrm{e}^{-\beta(T-s)}\right)\,ds\,,$$

for ϕ being the log-moment generating function of $L(1)$. Observe that $\phi_\beta(\theta)$ is well-defined for all $|\theta| \le 3$. In the proof of the convergence to the Black-76 formula, we will need the following simple result.

Lemma 3.3. *The function $f(x) = (1-\exp(-x))/x$ for $x \ge 0$ is decreasing to zero with $f(0) = 1$.*

Proof. By L'Hopital's rule we find $f(0) = 1$. Moreover,

$$f'(x) = \frac{(x+1)\mathrm{e}^{-x} - 1}{x^2}\,,$$

and since $x + 1 \le \mathrm{e}^x$ it holds that $(x+1)\mathrm{e}^{-x} - 1 \le 0$ and thus $f'(x) \le 0$. Letting $x \to \infty$, we see that $f(x) \to 0$. The Lemma holds. □

In the results below, the positive constant c will be generic and not necessarily refer to the same value. We have,

Lemma 3.4. *It holds, for $\tau \le T$,*

$$\sup_{x\ge 0}\left|\mathbb{E}\left[\Phi\left(d_2\left(x, \int_t^\tau e^{-\beta(T-s)}\,dL(s)\right)\right)\right] - \Phi(d_2(x,0))\right| \le ce^{-\beta(T-\tau)}$$

for a constant $c > 0$.

Proof. It holds,

$$\left|\mathbb{E}\left[\Phi\left(d_2\left(x,\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right)\right]-\Phi(d_2(x,0))\right| \\ \leq \mathbb{E}\left[\left|\Phi\left(d_2\left(x,\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right)-\Phi(d_2(x,0))\right|\right].$$

By the mean value theorem, there exists a random variable Z such that

$$\Phi\left(d_2\left(x,\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right)-\Phi(d_2(x,0))=\Phi'(Z)\left(d_2\left(x,\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)-d_2(x,0)\right).$$

But, from the definition of Φ,

$$\Phi'(Z)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-Z^2/2}\leq\frac{1}{\sqrt{2\pi}}<1\,.$$

Furthermore,

$$d_2\left(x,\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)=d_2(x,0)+\frac{1}{\sigma\sqrt{\tau-t}}\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\,.$$

We therefore find, after using Cauchy-Schwarz' inequality,

$$\begin{aligned}\left|\mathbb{E}\left[\Phi\left(d_2\left(x,\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right)\right]-\Phi(d_2(x,0))\right| \\ <\frac{1}{\sigma\sqrt{\tau-t}}\mathbb{E}\left[\left|\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right|\right] \\ \leq\frac{1}{\sigma\sqrt{\tau-t}}\mathbb{E}\left[\left(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)^2\right]^{1/2}.\end{aligned}$$

From basic probability theory, we find

$$\begin{aligned}\mathbb{E}\left[\left(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)^2\right]&=\frac{d^2}{d\theta^2}\mathrm{e}^{\phi_\beta(\theta)}\Big|_{\theta=0} \\ &=\phi_\beta''(0)+(\phi_\beta'(0))^2 \\ &=\phi''(0)\int_t^\tau \mathrm{e}^{-2\beta(T-s)}\,ds+(\phi'(0))^2(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,ds)^2 \\ &=\left(\phi''(0)\frac{1}{2\beta}(1-\mathrm{e}^{-2\beta(\tau-t)})+(\phi'(0))^2\frac{1}{\beta^2}(1-e^{-\beta(\tau-t)})^2\right)\mathrm{e}^{-2\beta(T-\tau)}\,,\end{aligned}$$

But then we have

$$\begin{aligned}\left|\mathbb{E}\left[\Phi\left(d_2\left(x,\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right)\right]-\Phi(d_2(x,0))\right| \\ \leq\frac{1}{\sigma}\left[\phi''(0)\frac{1-\mathrm{e}^{-2\beta(\tau-t)}}{2\beta(\tau-t)}+(\phi'(0))^2\frac{1}{\beta^2}\frac{(1-\mathrm{e}^{-\beta(\tau-t)})^2}{\tau-t}\right]^{\frac{1}{2}}\mathrm{e}^{-\beta(T-\tau)}\,.\end{aligned}$$

Since $1-\exp(-\beta(\tau-t))\leq 1$, we use Lemma 3.3 twice to conclude the proof. □

In our next Lemma, we estimate the difference between $\Phi(d_2(x,0))$ and $\Phi(d_2(x))$.

Lemma 3.5. *It holds, for $\tau\leq T$,*

$$\sup_{x\geq 0}|\Phi(d_2(x,0))-\Phi(d_2(x))|\leq ce^{-\beta(T-\tau)}\,,$$

for a constant $c>0$.

Proof. We have that

$$d_2(x,0) = d_2(x) - \frac{1}{\sigma\sqrt{\tau - t}} \int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\, ds\,.$$

But then, appealing to the mean value theorem,

$$|\Phi(d_2(x,0)) - \Phi(d_2(x))| \leq \frac{c}{\sigma\sqrt{\tau - t}} \left| \int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\, ds \right|\,.$$

We analyse the integral on the right-hand side in more detail. For notational simplicity, let $\gamma(s) = \exp(-\beta(T-s))$. By definition of the log-moment generating function

$$\int_t^\tau \phi(\gamma(s))\, ds = \int_t^\tau \int_{\mathbb{R}} \left\{ \mathrm{e}^{\gamma(s)z} - 1 - \gamma(s) z \mathbf{1}_{|z|<1} \right\} \ell(dz)\, ds\,.$$

We have for $|z| \geq 1$,

$$\begin{aligned}
|\mathrm{e}^{\gamma(s)z} - 1| &\leq \sum_{n=1}^\infty \frac{(\gamma(s)|z|)^n}{n!} \\
&= \gamma(s)|z| \sum_{n=1}^\infty \frac{(\gamma(s)|z|)^{n-1}}{n!} \\
&\leq \gamma(s)|z| \sum_{n=0}^\infty \frac{(\gamma(s)|z|)^n}{n!} \\
&= \gamma(s)|z|\mathrm{e}^{\gamma(s)|z|}\,.
\end{aligned}$$

If $|z| < 1$, the series representation of the exponential function gives

$$\begin{aligned}
|\mathrm{e}^{\gamma(s)z} - 1 - \gamma(s)z| &\leq \sum_{n=2}^\infty \frac{(\gamma(s)|z|)^n}{n!} \\
&\leq \gamma^2(s)|z|^2 \sum_{n=2}^\infty \frac{\gamma^{n-2}(s)}{n!} \\
&= \gamma^2(s)|z|^2\mathrm{e}^{\gamma(s)}\,.
\end{aligned}$$

Hence, using the definition of $\gamma(s)$,

$$\begin{aligned}
\int_{\mathbb{R}} |\mathrm{e}^{\gamma(s)z} - 1 - \gamma(s)z\mathbf{1}_{|z|<1}|\, \ell(dz) &\leq \gamma^2(s) \int_{|z|<1} z^2\, \ell(dz)\mathrm{e}^{\gamma(s)} + \gamma(s) \int_{|z|\geq 1} |z|\mathrm{e}^{\gamma(s)|z|}\, \ell(dz) \\
&\leq \mathrm{e}^{-2\beta(T-s)} \int_{|z|<1} z^2\, \ell(dz) + \mathrm{e}^{-\beta(T-s)} \int_{|z|\geq 1} \mathrm{e}^{2|z|}\, \ell(dz) \\
&\leq \mathrm{e}^{-\beta(T-s)} \left(\int_{|z|<1} z^2\, \ell(dz) + \int_{|z|\geq 1} \mathrm{e}^{2|z|}\, \ell(dz) \right)\,.
\end{aligned}$$

Therefore, there exists a constant $c > 0$ such that

$$\begin{aligned}
\left| \frac{1}{\sigma\sqrt{\tau - t}} \int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\, ds \right| &\leq \frac{c}{\sigma\sqrt{\tau - t}} \int_t^\tau \mathrm{e}^{-\beta(T-s)}\, ds \\
&= \frac{c}{\sigma\beta} \frac{1 - \mathrm{e}^{-\beta(\tau - t)}}{\sqrt{\tau - t}} \mathrm{e}^{-\beta(T-\tau)}\,.
\end{aligned}$$

The Lemma follows by invoking Lemma 3.3. □

We move on analysing the first term in our pricing formula in Prop. 3.2.

Lemma 3.6. *It holds, for $\tau \leq T$,*

$$\sup_{x\geq 0} \Bigg| \mathbb{E}\left[\exp\left(\int_t^\tau e^{-\beta(T-s)}\, dL(s) - \int_t^\tau \phi(e^{-\beta(T-s)})\, ds \right) \Phi\left(d_1\left(x, \int_t^\tau e^{-\beta(T-s)}\, dL(s) \right) \right) \right]$$
$$-\Phi(d_1(x,0)) \Bigg| \leq ce^{-\beta(T-\tau)}$$

for a constant $c > 0$.

Proof. Introduce the probability $\widetilde{Q}$ with Radon-Nikodym derivative

$$\frac{d\widetilde{Q}}{dQ} = \exp\left(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\, dL(s) - \int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\, ds\right).$$

This is an Esscher transform, turning the Lévy process L into a independent increment process (with time-dependent compensator measure, see Benth et al. [5]). The logarithmic-moment generating function of $\int_t^\tau \mathrm{e}^{-\beta(T-s)}\, dL(s)$ under $\widetilde{Q}$ will become

$$\begin{aligned}\phi_{\beta,\widetilde{Q}}(\theta) &= \ln \mathbb{E}_{\widetilde{Q}}\left[\exp\left(\theta \int_t^\tau \mathrm{e}^{-\beta(T-s)}\, dL(s)\right)\right] \\ &= \ln \mathbb{E}\left[\exp\left(\int_t^\tau \{(1+\theta)\mathrm{e}^{-\beta(T-s)}\}\, dL(s)\right)\right] - \int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\, ds \\ &= \int_t^\tau \{\phi((1+\theta)\mathrm{e}^{-\beta(T-s)}) - \phi(\mathrm{e}^{-\beta(T-s)})\}\, ds\,.\end{aligned}$$

Since

$$d_1(x,v) = d_1(x,0) + \frac{v}{\sigma\sqrt{\tau - t}}\,,$$

we estimate as in the proof of Lemma 3.4 to get

$$\begin{aligned}\Bigg|\mathbb{E}\left[\exp\left(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\, dL(s) - \int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\, ds\right)\Phi\left(d_1\left(x, \int_t^\tau \mathrm{e}^{-\beta(T-s)}\, dL(s)\right)\right)\right] \\ -\Phi(d_1(x,0))\Bigg| \leq \frac{1}{\sigma\sqrt{\tau-t}}\mathbb{E}_{\widetilde{Q}}\left[\left(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\, dL(s)\right)^2\right]^{1/2}.\end{aligned}$$

Again, from elementary probability theory we find

$$\mathbb{E}_{\widetilde{Q}}\left[\left(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\, dL(s)\right)^2\right] = \phi''_{\beta,\widetilde{Q}}(0) + (\phi'_{\beta,\widetilde{Q}}(0))^2\,.$$

From our definition of $\phi_{\beta,\widetilde{Q}}$, we find by appealing to the dominated convergence theorem,

$$\phi'_{\beta,\widetilde{Q}}(\theta) = \int_t^\tau \phi'((1+\theta)\mathrm{e}^{-\beta(T-s)})\mathrm{e}^{-\beta(T-s)}\, ds\,,$$

which implies

$$\phi'_{\beta,\widetilde{Q}}(0) = \int_t^\tau \phi'(\mathrm{e}^{-\beta(T-s)})\mathrm{e}^{-\beta(T-s)}\, ds\,.$$

Denoting $\gamma(s) = \exp(-\beta(T-s))$, it follows,

$$\begin{aligned}\phi'(\gamma(s)) &= \frac{d}{d\xi}\int_{\mathbb{R}} \{\mathrm{e}^{\xi z} - 1 - \xi z \mathbf{1}_{|z|<1}\}\, \ell(dz)\Big|_{\xi=\gamma(s)} \\ &= \int_{\mathbb{R}} \{z\mathrm{e}^{\gamma(s)z} - z\mathbf{1}_{|z|<1}\}\, \ell(dz) \\ &= \int_{|z|<1} z\{\mathrm{e}^{\gamma(s)z} - 1\}\, \ell(dz) + \int_{|z|\geq 1} z\mathrm{e}^{\gamma(s)z}\, \ell(dz)\,.\end{aligned}$$

As

$$|\mathrm{e}^{\gamma(s)z} - 1| \leq |z|\mathrm{e}^{\gamma(s)|z|} \leq |z|\mathrm{e}^1\,,$$

for $|z| < 1$, while for $|z| > 1$

$$|z|\mathrm{e}^{\gamma(s)|z|} \leq \mathrm{e}^{2|z|}\,,$$

it follows that

$$|\phi'(\gamma(s))| \leq \mathrm{e}^1 \int_{|z|<1} z^2\, \ell(dz) + \int_{|z|\geq 1} \mathrm{e}^{2|z|}\, \ell(dz) \leq c\,,$$

for a positive constant c. Similarly,

$$\phi''(\gamma(s)) = \frac{d}{d\xi}\int_{\mathbb{R}} z\{\mathrm{e}^{\xi z} - \mathbf{1}_{|z|<1}\}\,\ell(dz)\Big|_{\xi=\gamma(s)}$$
$$= \int_{\mathbb{R}} z^2\mathrm{e}^{\gamma(s)z}\,\ell(dz)\,.$$

As

$$z^2\mathrm{e}^{\gamma(s)|z|} \leq \mathrm{e}^1|z|^2\mathbf{1}_{|z|<1} + \mathbf{1}_{|z|\geq 1}\mathrm{e}^{3|z|}\,,$$

it follows from the condition on the Lévy measure in (2.4)

$$|\phi''(\mathrm{e}^{-\beta(T-s)})| \leq c$$

for some constant $c > 0$. Wrapping up the estimates, we are left with

$$\frac{1}{\sigma\sqrt{\tau-t}}\mathbb{E}_{\widetilde{Q}}\left[\left(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)^2\right]^{1/2}$$
$$\leq \frac{1}{\sigma}\left[c\,\frac{1}{\beta^2}\frac{(1-e^{-\beta(\tau-t)})^2}{\tau-t} + c\,\frac{1-\mathrm{e}^{-2\beta(\tau-t)}}{\tau-t}\right]^{\frac{1}{2}}\mathrm{e}^{-\beta(T-\tau)}$$

As the fractions in the last inequality by Lemma 3.3 are bounded, the Lemma is proven. □

We end with the Lemma,

Lemma 3.7. *It holds, for $\tau \leq T$,*

$$\sup_{x\geq 0}|\Phi(d_1(x,0)) - \Phi(d_1(x))| \leq ce^{-\beta(T-\tau)}\,,$$

for some constant $c > 0$.

Proof. Since

$$d_1(x,0) = d_1(x) - \frac{1}{\sigma\sqrt{\tau-t}}\int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\,ds\,,$$

the proof is identical to the proof of Lemma 3.5. □

We summarize our findings in the following theorem:

Theorem 3.8. *Suppose that $\tau \leq T$. Then it holds that*

$$\sup_{x\geq 0}|C(t,\tau,T,x) - C_{B76}(t,\tau,T,x)| \leq ce^{-\beta(T-\tau)}\,,$$

for some constant c.

Proof. Appealing to the triangle inequality and Lemmas 3.4-3.7 yield the result. □

By fixing τ, we see that the call option price $C(t,\tau,T,x)$ is converging uniformly to the Black-76 price as $T \to \infty$. The convergence is exponential with the rate β. We recall that β is the speed of mean reversion of the spike factor of the spot dynamics. Note also that tracing through the proofs of the Lemmas 3.4-3.7 we can find an expression for the constant c, and therefore we can find the maximal error between the Black-76 price and the "correct" price $C(t,\tau,T,x)$. However, the overall conclusion is that for options where the delivery time T is sufficiently bigger than τ, the call option price can be approximated with a high degree of accuracy by the Black-76 formula.

Recall that in electricity markets, forwards deliver over a period rather than at a fixed time T. A way of modelling such forwards is to introduce a dynamics $f(t,T^*)$, where $T^* \in (T_1,T_2)$ is some time in the delivery period $[T_1,T_2]$. A natural choice of T^* could be the middle point $T^* = (T_1+T_2)/2$. It is a well-known empirical fact that in electricity markets forward prices do not converge to the spot prices if one approaches delivery time. Choosing T^* as the mid point of the delivery period will take this into account (see Barndorff-Nielsen et al. [1]). In the electricity markets, many options have exercise time equal to the beginning of delivery T_1 of the underlying forward, e.g. $\tau = T_1$. If the delivery period is relatively long, we will have that τ is relatively far from T^*. Hence, for a reasonably strong mean reversion β of the spikes, options on forwards in electricity markets can be priced with a high degree of accuracy by the Black-76 formula.

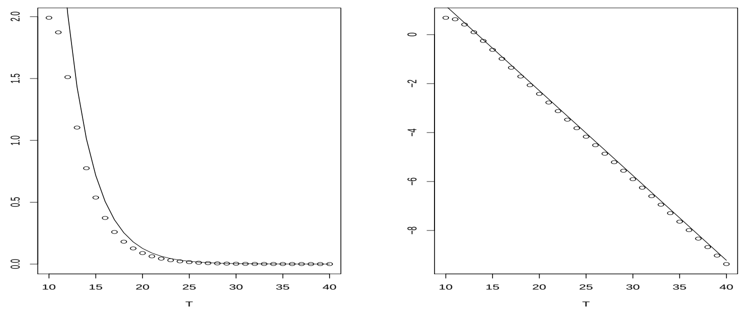

FIGURE 1. Difference of the option price to the Black-76 (left), and on log-scale (right). The solid line is the theoretical error estimate.

We illustrate our results with a numerical example. At $t = 0$, let the the excercise time of the option be in $\tau = 10$ days and consider forwards with delivery times in $T = 10, ..., 40$ days. Assume that the speed of mean reversion is $\beta = 0.3466$, which corresponds to a half life of two days. Such a mean reversion rate is not unreasonable for spikes in electricity markets (see e.g. Benth et al. [2]). We model directly under the pricing measure Q. Let L be a compound Poisson process that has an exponential jump size distribution with mean equal 0.5 and a jump intensity of 5 jumps per month. This is a rather high number of spikes, but could mimic the situation in winter months, say, in the Nordic electricity market NordPool. The volatility the Brownian motion is $\sigma = 0.0158$ which corresponds to 30% annually. For simplicity, we assume that there is no seasonality, that is $\Lambda(t) = 1$ and fix the initial value of the forward to $x = 100$. We look at options at the money and assume $K = 100$. Using the Black-76 formula in Prop. 3.1, we get $C_{B76} = 1.9931$. We evaluate the option price $C(0, 10, T, 100)$ as in Prop. 3.2 with Monte Carlo simulation. For this purpose, the stochastic integral in Prop. 3.2 is discretized with a simple Euler scheme on a daily time grid. The price differences as well as the logarithmic price differences are plottet in Figure 1 together with the corresponding error bound from Prop. 3.2 (solid lines). The exponential decay of the error is in line with our theoretical results. If we consider an electricity forward with a monthly delivery period of 30 days, that is, $T_2 - T_1 = 30$, which starts at $T_1 = 10$, this would correspond to a $T = 25$ days if we let $T^* = (T_1 + T_2)/2$. Looking at Figure 1 we see that at this time, the prices using Prop. 3.2 and Prop. 3.1 are already very close. In fact, we have that $C(0, 10, 25, 100) = 1.9337$, implying that Black-76 is mis-pricing by only 3 %.

3.1. **A transformed-based option pricing formula.** For the sake of completeness, we include here a pricing formula for $C(t, \tau, T, x)$ based on the Fourier transform and the characteristic function of $f(t, T)$.

Recall the price $C(t, \tau, T, x)$ in Prop. 3.2. Denote the first expectation by I_1 and the second by I_2. In Lemma 3.6 we changed probability from Q to $\widetilde{Q}$ to reach the expression

$$I_1 = \mathbb{E}_{\widetilde{Q}}\left[\Phi\left(d_1\left(x, \int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right)\right], \tag{3.4}$$

where the logarithmic cumulant function $\phi_{\beta,\widetilde{Q}}(\theta)$ of $\int_t^\tau \exp(-\beta(T-s))\,dL(s)$ with respect to $\widetilde{Q}$ is

$$\phi_{\beta,\widetilde{Q}}(\theta) = \int_t^\tau \{\phi((1+\theta)\mathrm{e}^{-\beta(T-s)}) - \phi(\mathrm{e}^{-\beta(T-s)})\}\,ds\,. \tag{3.5}$$

We now want to express the expectations I_1 and I_2 by Fourier transforms.

Letting $d(x,v)$ be a generic notation for $d_1(x,v)$ and $d_2(x,v)$, we find that $\Phi(d(x,v)) \to 1$ when $v \to \infty$ since $d(x,v) \to \infty$ when $v \to \infty$. On the other hand, as $d(x,v) \to -\infty$ when $v \to -\infty$, we find $\Phi(d(x,v)) \to 0$. Hence, the function $v \mapsto \Phi(d(x,v))$ is not in $L^1(\mathbb{R})$. However, by damping it using an exponential function, we get an expression which is integrable:

Lemma 3.9. *For any $\alpha > 0$, the function $v \mapsto \exp(-\alpha v)\Phi(d(x,v))$ is integrable on $\mathbb{R}$. Here $d(x,v)$ is generic for $d_i(x,v)$, $i = 1,2$.*

Proof. Since $0 \le \Phi(y) \le 1$, we have by Tonelli's theorem (see Folland [9])

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{e}^{-\alpha v}\Phi(d(x,v))\,dv &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\alpha v}\int_{-\infty}^{d(x,v)} \mathrm{e}^{-y^2/2}\,dy\,dv \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{-\alpha v}\mathbf{1}_{y\le d(x,v)}\,dv\,\mathrm{e}^{-y^2/2}\,dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\sigma\sqrt{\tau-t}(y-d(x,0))}^{\infty} \mathrm{e}^{-\alpha v}\,dv\,\mathrm{e}^{-y^2/2}\,dy \\ &= \frac{1}{\alpha\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2}y^2-\alpha\sigma\sqrt{\tau-t}(y-d(x,0))}\,dy < \infty\,.\end{aligned}$$

Hence, the Lemma follows. □

In the next Lemma we compute the Fourier transform of the function $v \mapsto \Phi_\alpha(d(x,v)) := \exp(-\alpha v)\Phi(d(x,v))$:

Lemma 3.10. *The Fourier transform of $v \mapsto \Phi_\alpha(d(x,v))$ is*

$$\widehat{\Phi}_\alpha(y) = \frac{1}{\alpha+\mathrm{i}y}\exp\left(\frac{1}{2}(\alpha+\mathrm{i}y)^2\sigma^2(\tau-t) + (\alpha+\mathrm{i}y)d(x,0)\sigma\sqrt{\tau-t}\right)\,.$$

Moreover, $\widehat{\Phi}_\alpha \in L^1(\mathbb{R})$.

Proof. By definition of the Fourier transform and the Fubini-Tonelli theorem (see Folland [9]), we find

$$\begin{aligned}\widehat{\Phi}_\alpha(y) &= \int_{\mathbb{R}} \mathrm{e}^{-\alpha v}\Phi(d(x,v))\mathrm{e}^{-\mathrm{i}vy}\,dv \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{-\infty}^{d(x,v)} \mathrm{e}^{-z^2/2}\,dz\,\mathrm{e}^{-(\alpha+\mathrm{i}y)v}\,dv \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{(z\le d_2(x,v))}\mathrm{e}^{-(\alpha+\mathrm{i}y)v}\,dv\,\mathrm{e}^{-z^2/2}\,dz \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\sigma\sqrt{\tau-t}(z-d_2(x,0))}^{\infty} \mathrm{e}^{-(\alpha+\mathrm{i}y)v}\,dv\,\mathrm{e}^{-z^2/2}\,dz \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\alpha+\mathrm{i}y}\int_{\mathbb{R}} \mathrm{e}^{-(\alpha+\mathrm{i}y)\sigma\sqrt{\tau-t}z}\mathrm{e}^{-z^2/2}\,dz\,\mathrm{e}^{d_2(x,0)\sigma\sqrt{\tau-t}(\alpha+\mathrm{i}y)} \\ &= \frac{1}{\alpha+\mathrm{i}y}\mathbb{E}\left[\mathrm{e}^{-(\alpha+\mathrm{i}y)\sigma\sqrt{\tau-t}Z}\right]\mathrm{e}^{d_2(x,0)\sigma\sqrt{\tau-t}(\alpha+\mathrm{i}y)}\,.\end{aligned}$$

Here, Z is a standard normally distributed random variable. This means that

$$\mathbb{E}\left[\mathrm{e}^{-(\alpha+\mathrm{i}y)\sigma\sqrt{\tau-t}Z}\right] = \mathrm{e}^{\frac{1}{2}(\alpha+\mathrm{i}y)^2\sigma^2(\tau-t)}\,.$$

This shows the Fourier transform of $\Phi_\alpha(x,v)$.

By taking absolute values, we find

$$|\widehat{\Phi}_\alpha(y)| = \frac{c}{\alpha^2+y^2}\mathrm{e}^{-\frac{1}{2}y^2\sigma^2(\tau-t)}\,,$$

for a constant c independent of y. This shows that $\widehat{\Phi}_\alpha$ is an integrable function on $\mathbb{R}$. The proof is complete. □

Appealing to the inverse Fourier transform (see Folland [9]), we have the relation

$$\Phi(d(x,v)) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}_{\alpha}(y)\mathrm{e}^{(\alpha+\mathrm{i}y)v}\,dy\,. \tag{3.6}$$

We apply this in order to express the call option price in terms of the Fourier transform of Φ_α and the characteristic function of L:

Proposition 3.11. *The call option price $C(t,\tau,T,x)$ in Prop. 3.2 can be expressed as*

$$\begin{aligned} C(t,\tau,T,x) &= \frac{x}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}_{1,\alpha}(y)\exp\left(\phi_{\beta,\widetilde{Q}}(\alpha+\mathrm{i}y)\right)\,dy \\ &\quad - \frac{K}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}_{2,\alpha}(y)\exp\left(\int_t^\tau \phi\left((\alpha+\mathrm{i}y)e^{-\beta(T-s)}\right)\,ds\right)\,dy \end{aligned}$$

for any $0<\alpha\leq 2$. We have introduced the notation $\widehat{\Phi}_{i,\alpha}$ to indicate that we use $d_i(x,v)$, $i=1,2$ as the function $d(x,v)$.

Proof. Using (3.6) it holds

$$\begin{aligned} \mathbb{E}\left[\Phi\left(d_2\left(x,\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right)\right] &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}_{2,\alpha}(y)\mathbb{E}\left[\exp\left((\alpha+\mathrm{i}y)\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right]\,dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}_{2,\alpha}(y)\exp\left(\int_t^\tau \phi\left((\alpha+\mathrm{i}y)\mathrm{e}^{-\beta(T-s)}\right)\,ds\right)\,dy\,. \end{aligned}$$

Note that we must have $\alpha\leq 3$ in order for this to be well-defined, according to the exponential integrability condition (see Theorem 25.17(iii) in Sato [14]). This shows the second term in the price.

For the first term, we use the expectation under the probability $\widetilde{Q}$ as in (3.4). Again using the inverse Fourier transform along with Fubini-Tonelli's theorem,

$$\begin{aligned} \mathbb{E}_{\widetilde{Q}}\left[\Phi\left(d_1\left(x,\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right)\right] &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}_{1,\alpha}(y)\mathbb{E}_{\widetilde{Q}}\left[\exp\left((\alpha+\mathrm{i}y)\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right]\,dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}_{1,\alpha}(y)\exp\left(\phi_{\beta,\widetilde{Q}}(\alpha+\mathrm{i}y)\right)\,dy\,. \end{aligned}$$

Note that $\phi_{\beta,\widetilde{Q}}(\alpha+\mathrm{i}y)$ is well-defined as long as $\alpha\leq 2$. This proves the first term, and the Proposition follows. □

We remark that the transformed-based pricing equation in the Proposition above lends itself to fast Fourier transform metods for numerical evaluation (see Eberlein et al. [8]).

4. Quadratic hedging of call options on forwards

We next consider hedging of the call option. Our market is incomplete, since the forward price dynamics is a jump-diffusion process. In this case there exists no self-financing portfolio in the underlying forward contract and a bank account replicating the option exactly. Instead, one must apply hedging strategies which minimizes, under some criterion, the hedging error. The hedging error is defined to be the difference between the terminal value of the hedging portfolio and the option, and we shall here look at hedges which minimize the variance, also called quadratic hedging. We refer the reader to Cont and Tankov [7] for a detailed discussion on incomplete markets and hedging, in particular quadratic hedging.

The next Proposition states the quadratic hedge position in the forward:

Proposition 4.1. *The quadratic hedge position $\psi(t)$ at time $t\leq\tau$ in the forward is given by*

$$\begin{aligned} \psi(t) &= \frac{\sigma^2}{\sigma^2+\int_{\mathbb{R}}(e^{ze^{-\beta(T-t)}}-1)^2\,\ell(dz)}C_x(t,\tau,T,f(t,T)) \\ &\quad + \frac{f^{-1}(t,T)\int_{\mathbb{R}}(e^{ze^{-\beta(T-t)}}-1)\left(C(t,\tau,T,f(t,T)e^{ze^{-\beta(T-t)}})-C(t,\tau,T,f(t,T))\right)\,\ell(dz)}{\sigma^2+\int_{\mathbb{R}}(e^{ze^{-\beta(T-t)}}-1)^2\,\ell(dz)}\,. \end{aligned}$$

Proof. First, let $\widetilde{C}(t,\tau,T,x)=\mathrm{e}^{-rt}C(t,\tau,T,x)$, the discounted option price. We know that the process $t\mapsto\widetilde{C}(t,\tau,T,x)$ is a martingale by the no-arbitrage pricing theory. Applying Itô's Formula for jump-diffusion, shows that

$$d\widetilde{C}(t,\tau,T,f(t,T))=\sigma f(t,T)\widetilde{C}_x(t,\tau,T,f(t,T))\,dW(t)\\ +\int_{\mathbb{R}}\left\{\widetilde{C}(t,\tau,T,f(t,T)\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}})-\widetilde{C}(t,\tau,T,f(t,T))\right\}\widetilde{N}(dt,dz)\,.$$

If we let $\widetilde{V}(t)=\mathrm{e}^{-rt}V(t)$ be the discounted value of a self-financing portfolio (which is a martingale as well), then

$$d\widetilde{V}(t)=\psi(t)\mathrm{e}^{-rt}\,df(t,T)=\psi(t)\mathrm{e}^{-rt}f(t-,T)\left\{\sigma\,dW(t)+\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)\,\widetilde{N}(dt,dz)\right\}\,.$$

Suppose that $V(0)=\widetilde{V}(0)=C(0,\tau,T,f(0,T))$. The hedging error is

$$\epsilon(\psi)=\widetilde{V}(\tau,\tau,T,f(\tau,T))-\widetilde{C}(\tau,\tau,T,f(\tau,T))\,,$$

for $\tau\leq T$. But then it follows by Itô isometry for stochastic integration (see Cont and Tankov [7])

$$\begin{aligned}\mathbb{E}[\epsilon^2(\psi)]=&\int_0^\tau\mathbb{E}\left[(\widetilde{C}_x(s,\tau,T,f(s,T))-\psi(s))^2\mathrm{e}^{-2rs}f^2(s,T)\right]\sigma^2\,ds\\ &\int_0^\tau\int_{\mathbb{R}}\mathbb{E}\left[\left(\widetilde{C}(s,\tau,T,f(s,T)\mathrm{e}^{z\mathrm{e}^{-\beta(T-s)}})-\widetilde{C}(s,\tau,T,f(s,\tau,T,f(s,T))\right.\right.\\ &\left.\left.-\psi(s)\mathrm{e}^{-rs}f(s,T)(\mathrm{e}^{z\mathrm{e}^{-\beta(T-s)}}-1)\right)^2\right]\ell(dz)\,ds\\ =&\int_0^\tau\mathrm{e}^{-2rs}\mathbb{E}\left[f^2(s,T)\left\{\sigma^2\left(C_x(s,\tau,T,f(s,T))-\psi(s)\right)^2+\right.\right.\\ &+\int_{\mathbb{R}}\left(\frac{C(s,\tau,T,f(s,T)\mathrm{e}^{z\mathrm{e}^{-\beta(T-s)}})-C(s,\tau,T,f(s,T))}{f(s,T)}\right.\\ &\left.\left.\left.-\psi(s)(\mathrm{e}^{z\mathrm{e}^{-\beta(T-s)}}-1)\right)^2\ell(dz)\right\}\right]ds\,.\end{aligned}$$

The first order condition for the minimizer of this quadratic expression solves

$$\psi(t)\left(\sigma^2+\int_{\mathbb{R}}\left(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1\right)^2\ell(dz)\right)=\sigma^2C_x(t,\tau,T,f(t,T))\\ +f^{-1}(t,T)\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)\left(C(t,\tau,T,f(t,T)\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}})-C(t,\tau,T,f(t,T))\right)\ell(dz)$$

Hence, the Proposition follows. □

Our goal now is to show that this quadratic hedge converges to the delta hedging strategy $C_x(t,\tau,T,x)$ of the Black-76 call. To have a more suggestive notation, we let

$$C(t,x;\beta):=C(t,\tau,T,x)$$

and we recall from from theorem 3.8 that

$$\lim_{\beta\downarrow 0}C(t,x;\beta)=C_{\mathrm{B76}}(t,x)\,,$$

with the obvious meaning of the short-hand notation $C_{\mathrm{B76}}(t,x)$. The Black-76 formula is the price of a call in a complete market, and the hedge position in the forward is given by the derivative of the call price with respect to the forward, $C_{\mathrm{B76},x}(t,x)$ (see Cont and Tankov [7], say). We show, in a sequence of Lemmas, that

$$\psi(t)\to C_{\mathrm{B76},x}(t,x)\,,$$

when $T-t\to\infty$. Moreover, we show that the convergence is uniform with an exponential rate given by β.

First, we recall the delta hedge in the Black-76 market:

Proposition 4.2. *The delta hedge of Black-76 is*

$$C_{B76,x}(t,x)=\Phi(d_1(x))\,,$$

with $d_1(x)$ defined in Prop. 3.1.

Proof. This is a straightforward application of the result in Prop. 3.1. □

The next Proposition shows that the derivative of $C(t,x;\beta)$ has the same shape as the delta hedge in the Black-76 market:

Proposition 4.3. *For every $t\leq\tau\leq T$ and $\beta>0$ it holds*

$$C_x(t,x;\beta)=\mathbb{E}\left[\exp\left(\int_t^\tau e^{-\beta(T-s)}\,dL(s)-\int_t^\tau\phi(e^{-\beta(T-s)})\,ds\right)\Phi\left(d_1\left(x,\int_t^\tau\phi(e^{-\beta(T-s)})\,dL(s)\right)\right)\right]\,,$$

where $d_1(x,v)$ is defined in Prop. 3.2.

Proof. A direct derivation of the expression in Prop. 3.2 yields,

$$\begin{aligned}C_x(t,x;\beta)&=\mathbb{E}\left[\mathrm{e}^{Z-\int_t^\tau\phi(\mathrm{e}^{-\beta(T-s)})\,ds}\Phi\left(d_1(x,Z)\right)\right]\\&\quad+x\mathbb{E}\left[\mathrm{e}^{Z-\int_t^\tau\phi(\mathrm{e}^{-\beta(T-s)})\,ds}\Phi'\left(d_1(x,Z)\right)\frac{\partial d_1(x,Z)}{\partial x}\right]\\&\quad-K\mathbb{E}\left[\Phi'\left(d_2(x,Z)\right)\frac{\partial d_2(x,Z)}{\partial x}\right]\,,\end{aligned}$$

with $Z=\int_t^\tau\exp(-\beta(T-s))\,dL(s)$. We focus on the last two terms, which we show are adding up to zero. From the definitions of $d_i(x,v)$, $i=1,2$,

$$\begin{aligned}\frac{\partial d_1(x,v)}{\partial x}&=\frac{\partial d_2(x,v)}{\partial x}\,,\\\frac{\partial d_1(x,v)}{\partial x}&=\frac{1}{x}\frac{1}{\sigma\sqrt{\tau-t}}\,.\end{aligned}$$

Hence,

$$\begin{aligned}&x\mathbb{E}\left[\mathrm{e}^{Z-\int_t^\tau\phi(\mathrm{e}^{-\beta(T-s)})\,ds}\Phi'\left(d_1(x,Z)\right)\frac{1}{x\sigma\sqrt{\tau-t}}\right]-K\mathbb{E}\left[\Phi'\left(d_2(x,Z)\right)\frac{1}{x\sigma\sqrt{\tau-t}}\right]\\&\quad=\frac{1}{\sigma\sqrt{\tau-t}}\mathbb{E}\left[\mathrm{e}^{Z-\int_t^\tau\phi(\mathrm{e}^{-\beta(T-s)})\,ds}\Phi'\left(d_1(x,Z)\right)-\frac{K}{x}\Phi'\left(d_2(x,Z)\right)\right]\,.\end{aligned}$$

As Φ is the cumulative distribution function of a standard normal variable, we find from the definition of $d_1(x,v)$ that

$$\Phi'\left(d_2(x,Z)\right)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}d_2^2(x,Z)\right)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(d_1(x,Z)+\sigma\sqrt{\tau-t})^2\right)\,.$$

But since

$$d_1(x,Z)\sigma\sqrt{\tau-t}=\ln x-\ln K+Z-\int_t^\tau\phi(\mathrm{e}^{-\beta(T-s)})\,ds-\frac{1}{2}\sigma^2(\tau-t)\,,$$

we have

$$\mathrm{e}^{Z-\int_t^\tau\phi(\mathrm{e}^{-\beta(T-s)})\,ds}\Phi'\left(d_1(x,Z)\right)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}d_2^2(x,Z)\right)\frac{K}{x}\,.$$

This concludes the proof of the Proposition. □

In the first lemma, we study the convergence of the "variance" term from the jumps:

Lemma 4.4. *For $t\leq T$, it holds*

$$\int_{\mathbb{R}}(e^{ze^{-\beta(T-t)}}-1)^2\,\ell(dz)\leq ce^{-2\beta(T-t)}\,,$$

for a constant $c>0$.

Proof. Although we have implicitly estimated this convergence in the Lemmas of the previous Section, we spell it out here for the convenience of the reader. For any positive constant $k \leq 1$ we have

$$|\mathrm{e}^{kz}-1| \leq k|z|\sum_{n=1}^{\infty}\frac{k^{n-1}|z|^{n-1}}{n!} \leq k|z|\sum_{n=0}^{\infty}\frac{|z|^n}{n!} = k|z|\mathrm{e}^{|z|}\,.$$

But then

$$\begin{aligned}\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)^2\,\ell(dz) &\leq \int_{\mathbb{R}}|z|^2\mathrm{e}^{2|z|}\,\ell(dz)\mathrm{e}^{-2\beta(T-t)}\\ &\leq \left\{\int_{|z|\leq 1} z^2\,\ell(dz)\mathrm{e}^2+\int_{|z|>1} z^2\mathrm{e}^{2|z|}\,\ell(dz)\right\}\mathrm{e}^{-2\beta(T-t)}\,.\end{aligned}$$

The Lemma follows from the exponential moment condition on $\ell(dz)$ and the condition that ℓ is a Lévy measure. □

A convenient property of the option price is that it is uniformly Lipschitz, as the next Lemma shows.

Lemma 4.5. *For every $t \leq \tau \leq t$ and $\beta > 0$, we have*

$$|C(t,x;\beta)-C(t,y;\beta)| \leq |x-y|\,,$$

for all $x, y \geq 0$.

Proof. By the mean-value theorem we find

$$|C(t,x;\beta)-C(t,y;\beta)| = |C_x(t,z;\beta)||x-y|\,,$$

for some $z \geq 0$. From Prop. 4.3 it follows

$$\begin{aligned}|C_x(t,z;\beta)| &= \mathbb{E}\left[\exp\left(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)-\int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\,ds\right)\Phi\left(d_1\left(z,\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)\right)\right)\right]\\ &\leq \mathbb{E}\left[\exp\left(\int_t^\tau \mathrm{e}^{-\beta(T-s)}\,dL(s)-\int_t^\tau \phi(\mathrm{e}^{-\beta(T-s)})\,ds\right)\right]\\ &= 1\,.\end{aligned}$$

The Lemma follows. □

We present our convergence result on the quadratic hedge in the next Theorem:

Theorem 4.6. *For $t \leq \tau \leq T$ it holds that*

$$\sup_{x\geq 0}|\psi(t)-C_{B76,x}(t,x)| \leq ce^{-\beta(T-\tau)}\,,$$

for a positive constant c.

Proof. By the triangle inequality it holds

$$\begin{aligned}|\psi(t)-C_{\mathrm{B76},x}(t,x)| \leq &\left|\frac{\sigma^2}{\sigma^2+\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)^2\,\ell(dz)}C_x(t,x;\beta)-C_{\mathrm{B76},x}(t,x)\right|\\ &+\frac{1}{\sigma^2+\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)^2\,\ell(dz)}\\ &\times\int_{\mathbb{R}}|\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1|\left|\frac{C(t,x\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}};\beta)-C(t,x;\beta)}{x}\right|\,\ell(dz)\\ \leq &|C_x(t,x;\beta)-C_{\mathrm{B76},x}(t,x)|+\frac{\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)^2\,\ell(dz)}{\sigma^2+\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)^2\,\ell(dz)}C_{\mathrm{B76},x}(t,x)\\ &+\frac{\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)^2\,\ell(dz)}{\sigma^2+\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)^2\,\ell(dz)}\,.\end{aligned}$$

In the last inequality we applied the Lipschitz continuity of C in Lemma 4.5. But from Prop. 4.2 we have that $C_{\mathrm{B76},x}(t,x) \leq 1$. Moreover,

$$\frac{\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)^2\,\ell(dz)}{\sigma^2+\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)^2\,\ell(dz)} \leq \frac{1}{\sigma^2}\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)^2\,\ell(dz)\,.$$

This implies that

$$|\psi(t)-C_{\mathrm{B76},x}(t,x)| \leq |C_x(t,x;\beta)-C_{\mathrm{B76},x}(t,x)| + \frac{2}{\sigma^2}\int_{\mathbb{R}}(\mathrm{e}^{z\mathrm{e}^{-\beta(T-t)}}-1)^2\,\ell(dz)\,.$$

Invoking Prop. 4.3, and using Lemmas 3.6-3.7 the first term on the right hand side can be bounded uniformly in x by $\exp(-\beta(T-\tau))$. Hence, we conclude the result by appealing to Lemma 4.4. □

Not surprisingly, the convergence rate of the delta hedge is equal to the one for the prices. Thus, when $\beta(T-\tau)$ is sufficiently big, the quadratic hedge of the call option will be approximately equal to the Black-76 delta hedge. Again, referring back to electricity forwards, we may have this situation when the delivery period $[T_1, T_2]$ is relatively long compared with the speed of mean reversion β, letting $T = (T_1+T_2)/2$.

5. Conclusion

Based on a generalization of the popular two-factor spot price model of Gibson and Schwartz [12], we show that call options written on forwards can be approximated by the Black-76 price in many situations. The logarithmic spot price dynamics consists of a non-stationary drifted Brownian motion factor, and a stationary factor modelled as a Lévy-driven Ornstein-Uhlenbeck process. The forward price becomes reasonably analytic under this spot model, and we derive the price of call options based on Fourier methods.

It is demonstrated that the option prices converge exponentially to the Black-76 price in terms of the speed of mean-reversion of the stationary factor in the spot price and the time left to maturity of the forward from the exercise time of the call. In many power markets, the stationary factor has a rather high speed of mean-reversion as this is modelling the spiky behaviour of spot prices. For options with exercise time relatively far from the delivery time of the forwards, the price will therefore be approximately given by the Black-76 formula. On the other hand, if the difference between time of delivery and exercise of option is small, the option price may be significantly far away from Black-76, unless the speed of mean-reversion is huge.

Typically, in gas and electricity, forwards deliver over a specified period like a month. In our framework we suggest to take delivery period forwards into account by assuming their dynamics being given by a forward delivering in the middle of the delivery period. Combining this approach with a typically high speed of mean-reversion, we can conclude that call options on electricity and gas forwards may be priced reasonably accurately by the Black-76 formula. In other words, we may completely ignore the spikes and non-Gaussian effects in the pricing, as these are "killed" by the delivery period of the forward. A numerical example further argue for this.

As our model for the spot and forward prices leads to an incomplete market, we cannot hedge the call option. However, the quadratic hedging strategy minimizing the L^2-distance between the call payoff and a portfolio in the underlying forward can be derived in terms of the option price. It is shown that the quadratic hedge can be approximated by the delta-hedge from Black-76. Not surprisingly, the hedge tends exponentially to the Black-76 delta hedge at the same rate as the option price.

There exist several interesting extensions of our results that could be worthwhile pursuing. For example, empirical studies of spot prices of electricity suggest that the stationary factor can be better modelled using a more general continuous time autoregressive moving average dynamics than the "AR(1)" Ornstein-Uhlenbeck process (see Garcia, Klüppelberg and Müller [10]). Another extension is to let the non-stationary factor be non-Gaussian, which is relevant in electricity (see e.g. Benth et al. [3]). Of course, such a spot model would not yield a convergence of option prices to the Black-76 formula as this rests on the Brownian motion driving the non-stationary part. A completely different path to follow is to check different hedging strategies than the quadratic one to analyse a possible convergence to the delta hedge of Black-76. This would lead into a different set-up for pricing and hedging of the options.

From an empirical point of view, it would be interesting to check our results with real option data in various markets. An immediate problem with such a study is that the liquidity in many energy option markets is rather low. Also, as we have mentioned in the above paragraphs, more sophisticated spot and forward models may be needed to reach firm conclusions. In any case, our analysis points towards the fact that the non-stationary factor is decisive for pricing and hedging options on forwards in energy markets.

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