ON THE SPEED TOWARDS THE MEAN FOR CARMA PROCESSES WITH APPLICATIONS TO ENERGY MARKETS

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1. Introduction

In energy, it is usual to model the spot price dynamics using a (subordination of) stationary processes. Typically, one uses a sum of Ornstein-Uhlenbeck processes

$$dX(t) = -\alpha X(t) + dL(t)$$

where α is called the speed of mean-reversion and $L(t)$ is a Lévy process. More recently, CARMA models have been applied succesfully to model such spot prices. It is of interest to understand the pathwise properties of such processes namely in order to understand if they reflect the observed dynamics, like for example the mean reversion in the case of a spike in electricity and gas markets. We observe usually a strong puch back after the price has experienced a large upward jump, in order to have a spike. Meanwhile, the base signal behaves much smoother in the sense that the small price variations are reverting at a slower rate. We want to understand the speed of mean-reversion of general CARMA processes, which is the purpose of this paper. The speed is measured through the concept of half-life, introduced by Clewlow and Strickland [1] for Ornstein-Uhlenbeck processes driven by Brownian motion.

We derive prices in risk neutral markets for CAT-type futures contract using general CARMA process modelling the temperature evolution. In Benth et al. [2], a continuous-time AR model which is a subclass of CARMA processes has been successfully applied in modelling the temperature behaviour towards pricing the temperature futures.

2. CARMA model with stochastic volatility

We study the notion of *half-life* for a Lévy-driven continuous-time autoregressive moving-average (CARMA) process with stochastic volatility defined as

$$Y(t) = \mathbf{b}'\mathbf{X}(t)\,, \tag{2.1}$$

with $\mathbf{X}(t)$ being the solution of the stochastic differential equation

$$d\mathbf{X}(t) = A\mathbf{X}(t)\,dt + \mathbf{e}_p\sigma(t)dL(t)\,. \tag{2.2}$$

Here, $L(t)$ is a Lévy process and A is the $p \times p$-matrix

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$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & \cdots & -\alpha_1 \end{bmatrix} \tag{2.3}$$

where α_k for $k = 1, \ldots, p$ are positive constants. Further, the vector $\mathbf{b}$ has elements b_j, $j = 0, \ldots, p-1$ with $b_j = 0, q < j < p$ and $b_q = 1$. The analytical solution of $\mathbf{X}(s)$ for given $\mathbf{X}(t)$, $s \geq t$ is

$$\mathbf{X}(s) = \exp(A(s-t))\mathbf{X}(t) + \int_t^s \exp(A(s-u))\mathbf{e}_p\sigma(u)\,dL(u) \tag{2.4}$$

where $\mathbf{e}_k$ is the kth unit vector in $\mathbb{R}^p$, $k = 1, \ldots, p$. The integral is defined as the L^2 limit of approximating Riemann-Stieltjes sums.

We choose to have a class of stochastic volatility process given by Barndorff-Nielsen and Shephard [3]. Let

$$V(t) = \sigma^2(t), \tag{2.5}$$

be the squared volatility which is a nonnegative stationary Lévy-driven Ornstein-Uhlenbeck process obtained by solving the following SDE

$$dV(t) = -\lambda V(t)dt + dZ(t). \tag{2.6}$$

A constant $\lambda > 0$ measures the speed of mean reversion exclusively for volatility process $V(t)$. This process is driven by the so-called *subordinator* $Z(t)$ which is independent of $L(t)$ and has an increasing path.

3. Half-life for a CARMA stochastic volatility model

We define the half-life of the CARMA process to be the time it takes before the (average) price is back to half of its distance to the long term mean. In mathematical terms, the half-life is defined as the time $\tau > t$ such that (assumed to be a stopping time)

$$\mathbb{E}\left[Y(\tau) - \mu \,|\, \mathcal{F}_t\right] = \frac{1}{2}(Y(t) - \mu)\,. \tag{3.1}$$

We see that τ will depend on current time t. Moreover, from the analytical solution of $\mathbf{X}(s)$ given $\mathbf{X}(t)$ it wil be a function of $\mathbf{X}(t)$ as well. Inserting the known expression from $\mathbf{X}(\tau)$ and conditioning we find the equation

$$\mathbf{b}'\exp(A(\tau-t))\mathbf{X}(t) + \phi_L'(0)\mathbf{b}'A^{-1}\left\{\exp(A(\tau-t)) - I\right\}\mathbf{e}_p - \mu = \frac{1}{2}\mathbf{b}'\mathbf{X}(t) - \frac{1}{2}\mu\,,$$

or, equivalently,

$$\mathbf{b}'\left\{\exp(A(\tau-t)) - \frac{1}{2}I\right\}\mathbf{X}(t) = -\phi_L'(0)\mathbf{b}'A^{-1}\exp(A(\tau-t))\mathbf{e}_p - \frac{1}{2}\mu \tag{3.2}$$

This is the equation that needs to be solved in order to find $\tau := \tau(t, \mathbf{X}(t))$.

3.1. **Lévy driven OU processes.** Consider now the special case of a Lévy driven OU process $Y(t)$, that is, $p=1, q=0$. Note that now $Y(t)=X(t)$, and $A=-\alpha_1 := \alpha, \mathbf{b}=1$. We find that the equation (3.2) simplifies to

$$\left\{ \mathrm{e}^{-\alpha(\tau-t)} - \frac{1}{2} \right\} Y(t) = \mu \left\{ \mathrm{e}^{-\alpha(\tau-t)} - \frac{1}{2} \right\}$$

or,

$$\mathrm{e}^{-\alpha(\tau-t)} \left\{ Y(t) - \mu \right\} = \frac{1}{2} \left\{ Y(t) - \mu \right\} .$$

Hence, we find that the half-life is

$$\tau = t + \frac{\ln(2)}{\alpha} .$$

which is the known solution from Clewlow and Strickland for a OU process driven by a Brownian motion. We see that the half-life in this case is not state-dependent, and it will on average always take $\ln(2)/\alpha$ time to revert back half the way from a deviation from the mean. This is irrespective of the driving Lévy process.

3.2. **CAR(p)-processes.** Let $p>1$ and $q=0$ be the case, when there is no moving average part in the dynamics. Assume the Lévy process has mean zero, meaning that $\phi_L'(0)=0$. The equation (3.2) simplifies to

$$\mathbf{b}' \left\{ \exp(A(\tau-t)) - \frac{1}{2} I \right\} \mathbf{X}(t) = 0. \tag{3.3}$$

This is corresponds to the state-dependent half life and to determine the half life in this implicit form is not a direct obtainable task. However, one can still find the estimated half-life by doing Monte Carlo simulation.

4. Pricing of temperature futures

We now turn to pricing the temperatures future based on Cumulative Average Temperature (CAT) index which is defined as the accumulated average temperature over a measurement period $[\tau_1, \tau_2]$, or it can be simply expressed as

$$\sum_{t=\tau_1}^{\tau_2} T(t). \tag{4.1}$$

Every CAT contract are settled at time τ_2, that is the last time of any measurement period for a contract entered at time $t \le \tau_1$. Since we can enter the contract for free, then the arbitrage theory holds where the CAT futures price can be derived from equation,

$$\mathrm{e}^{-r(\tau_2-t)} \mathbb{E}_Q \left[\int_{\tau_1}^{\tau_2} T(s) ds - F(t, \tau_1, \tau_2) | \mathcal{F}_t \right] = 0,$$

where r is a constant risk-free rate of return and Q is a risk-neutral probability. Thus, the arbitrage-free dynamics for CAT futures price is explicitly defined as

$$F(t,\tau_1,\tau_2)=\mathbb{E}_Q\left[\int_{\tau_1}^{\tau_2}T(s)ds|\mathcal{F}_t\right]. \tag{4.2}$$

To model futures price in risk-neutral market, one may introduces a change of measure in the dynamics for instance by Esscher transform. We denote

$$M(u)=L(u)-\mathbb{E}_Q[L(1)]\,u, \tag{4.3}$$

for $t\le u\le s$ which has mean zero. Under this transformation,

$$\begin{aligned}\mathbb{E}_Q\left[\int_t^s \mathbf{b}'\mathrm{e}^{A(s-u)}\mathbf{e}_p\sigma(u)dM(u)\right] &= 0,\\ \mathbb{E}_Q\left[\int_t^s \mathbf{b}'\mathrm{e}^{A(s-u)}\mathbf{e}_p\sigma(u)dL(u)\right] &= \int_t^s \mathbf{b}'\mathrm{e}^{A(s-u)}\mathbf{e}_p\sigma(u)\mathbb{E}_Q\left[L(1)\right]du,\\ &= \mathbb{E}_Q\left[L(1)\right]\int_t^s \mathbf{b}'\mathrm{e}^{A(s-u)}\mathbf{e}_p\sigma(u)du.\end{aligned} \tag{4.4}$$

Hence, the mean of CARMA process under risk neutral probability is

$$\mathbb{E}_Q\left[Y(s)|\mathcal{F}_t\right]=\mathbf{b}'\mathrm{e}^{A(s-t)}\mathbf{X}(t)+\mathbb{E}_Q\left[L(1)\right]\int_t^s \mathbf{b}'\mathrm{e}^{A(s-u)}\mathbf{e}_p\sigma(u)du. \tag{4.5}$$

We formulate the CAT futures price in the next proposition.

Proposition 4.1. *The CAT futures price at time t for $0\le t\le\tau_1<\tau_2$ and predetermined measurement period $[\tau_1,\tau_2]$ is given by*

$$\begin{aligned}F(t,\tau_1,\tau_2)=&\int_{\tau_1}^{\tau_2}S(s)ds+\mathbf{f}(t,\tau_1,\tau_2)\mathbf{X}(t)\\ &+\mathbb{E}_Q\left[L(1)\right]\int_t^{\tau_2}\mathbf{f}(u,\tau_1,\tau_2)\mathbf{e}_p\sigma(u)du\\ &+\mathbb{E}_Q\left[L(1)\right]\int_{\tau_1}^{\tau_2}\mathbf{b}'A^{-1}\left\{\mathrm{e}^{A(\tau_2)}-I_{p\times p}\right\}\mathbf{e}_p\sigma(u)du.\end{aligned} \tag{4.6}$$

where $\mathbf{f}(t,\tau_1,\tau_2)=\mathbf{b}'A^{-1}\{\exp(A(\tau_2-t))-\exp(A(\tau_1-t))\}$.

Proof.

$$F(t,\tau_1,\tau_2)=\int_{\tau_1}^{\tau_2}S(s)ds+\mathbb{E}_Q\left[\int_{\tau_1}^{\tau_2}Y(s)ds|\mathcal{F}_t\right].$$

$$\begin{aligned}\mathbb{E}_\theta\left[\int_{\tau_1}^{\tau_2}Y(s)ds|\mathcal{F}_t\right]&=\int_{\tau_1}^{\tau_2}\mathbb{E}_\theta\left[Y(s)|\mathcal{F}_t\right]ds\\ &=\int_{\tau_1}^{\tau_2}\mathbf{b}'\exp(A(s-t))ds\mathbf{X}(t)+\mathbb{E}_Q\left[L(1)\right]\int_{\tau_1}^{\tau_2}\int_t^s\mathbf{b}'\mathrm{e}^{A(s-u)}\mathbf{e}_p\sigma(u)duds.\end{aligned}$$

Here we use Fubini-Tonelli theorem to calculate the second integral,

$$
\begin{aligned}
\int_{\tau_1}^{\tau_2}\int_t^s \mathbf{b}'\exp(A(s-u))\mathbf{e}_p\sigma(u)du\,ds.& \\
&= \int_{\tau_1}^{\tau_2}\int_t^{\tau_2} \mathbf{1}_{[t,s]}\mathbf{b}'\exp(A(s-u))\mathbf{e}_p\sigma(u)du\,ds. \\
&= \int_t^{\tau_2}\int_{\tau_1}^{\tau_2} \mathbf{1}_{[t,s]}\mathbf{b}'\exp(A(s-u))\mathbf{e}_p\sigma(u)ds\,du. \\
&= \int_t^{\tau_1}\int_{\tau_1}^{\tau_2} \mathbf{b}'\exp(A(s-u))\mathbf{e}_p\sigma(u)ds\,du. \\
&\quad + \int_{\tau_1}^{\tau_2}\int_u^{\tau_2} \mathbf{b}'\exp(A(s-u))\mathbf{e}_p\sigma(u)ds\,du. \\
&= \int_t^{\tau_1} \mathbf{f}(u,\tau_1,\tau_2)\mathbf{e}_p\sigma(u)du \\
&\quad + \int_{\tau_1}^{\tau_2} \mathbf{b}'A^{-1}\{\exp(A(\tau_2-u)) - I_{p\times p}\}\mathbf{e}_p\sigma(u)du
\end{aligned}
$$

The proposition follows as we insert the result from integration and rearranging the terms. □

We should emphasize here the effect of duration between t and the time where measurement period start. In the study of seasonal futures contract by Dorfleitner and Wemmer [4] found that the price was likely to be constant for a certain time t before fluctuating at least one week before measurement. This was discussed in Benth and Šaltytė Benth [5] which leads to a general assumption that is when t is far away from the start, or we can simply say that the waiting period is too long, then the CAT futures price will be *essentially* constant as a reflection of mean reversion. As a consequence, the level of the price is setting by the seasonal function and a few deterministic terms. Our aim here is to find out the time when the prices being constant and this could be measured as the volalitily of the price dynamics being less than a certain fraction of the futures price. If the time t is too far from the measurement starts, then $\mathrm{e}^{A(\tau_1-t)} \to 0$.

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