Correlation as a pricing factor for oil derivatives

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This version: June 1st, 2012

This abstract contains color figures and is best printed using a color printer.

JEL Classification: G12,G13

Keywords: Oil; convenience yield; heteroskedasticity; dynamic correlation.

Abstract proposition for the 2012 Energy and Finance conference, Trondheim (Norway)

The goal of our paper is to show how correlation between convenience yield and commodity spot price must be and can be integrated to valuate commodity derivatives. This incorporation can be done in addition to usual factors: market prices of risk (Casassus and Collin-Dufresne, 2005) and/or stochastic volatility (Nielsen and Schwartz, 2004; Richter and Sørensen, 2006). While not discussed in this paper for ease of presentation, we could also take into account jumps (Hilliard and Reis, 1998; Koekebakker and Lien, 2004) in addition to stochastic correlation. Indeed, Cuchiero et al. (2011) derive necessary and sufficient conditions for the existence of affine-jump matrix valued processes. Their framework enables the modeling of the above cited stylized facts while keeping the usual tractability of the affine framework to price derivatives as well as to estimate their parameters.

Heteroskedasticity is a feature of storable commodities (Nielsen and Schwartz, 2004; Liu and Tang, 2011). Indeed, the theory of storage predicts that shocks will have (much) more

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important impacts on convenience yield and spot price during “stockouts” periods (see, e.g., Routledge et al. 2000). In the same vein, correlation between the spot price and the convenience yield is also time varying: as explained by Brennan (1991) the positive correlation between spot price and convenience yield stems from the fact that these two quantities depend on the level of inventory. As a consequence, correlation is supposed to be high when the level of inventory is an important determinant of commodity prices, i.e., in stockouts period. Routledge et al. (2000) explains this feature formally in an economic model. We confirm in Figures 1 and 2 the heteroskadicity of the convenience yield and the spot price as well as the time varying feature of the correlation between these two factors for the case of oil:

Figure 1. Oil spot price and convenience yield from 16/02/2000 to 16/04/2002

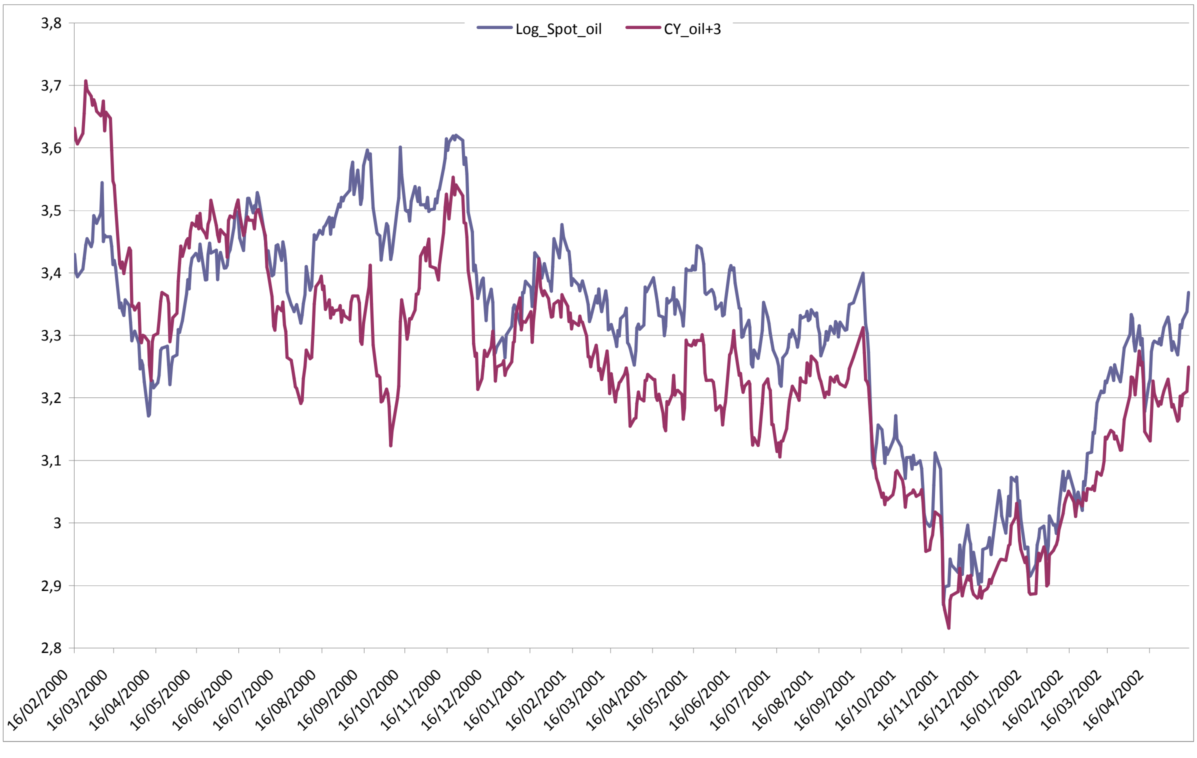

Figure 1. Oil spot price and convenience yield inferred from the model of Casassus and Collin-Dufresne (2005) as a function of time from 16/02/2000 to 16/04/2002. The spot price is displayed in logarithm while the convenience yield is in units of the spot price. To compare the dynamics of the (log) spot price and of the convenience yield, we display the convenience yield plus 3.

In the case of oil, both the spot price and the convenience yield are latent factors. As a consequence, a model of the term structure of futures price is needed to infer these two variables. We use the model of Casassus and Collin-Dufresne (2005), which takes into account important features of commodity date: time varying market price of risk and stochastic convenience yield. However, and of interest of our presentation, their model does not consider the heteroskedasticity of the convenience yield and the spot price, nor the time varying feature of the correlation. We choose a sample from the 16/02/2000 to the 16/04/2002. Other periods of time demonstrate a similar pattern.

Figure 2. Volatilities and correlation of the Oil spot price and convenience yield from 16/02/2000 to 16/04/2002

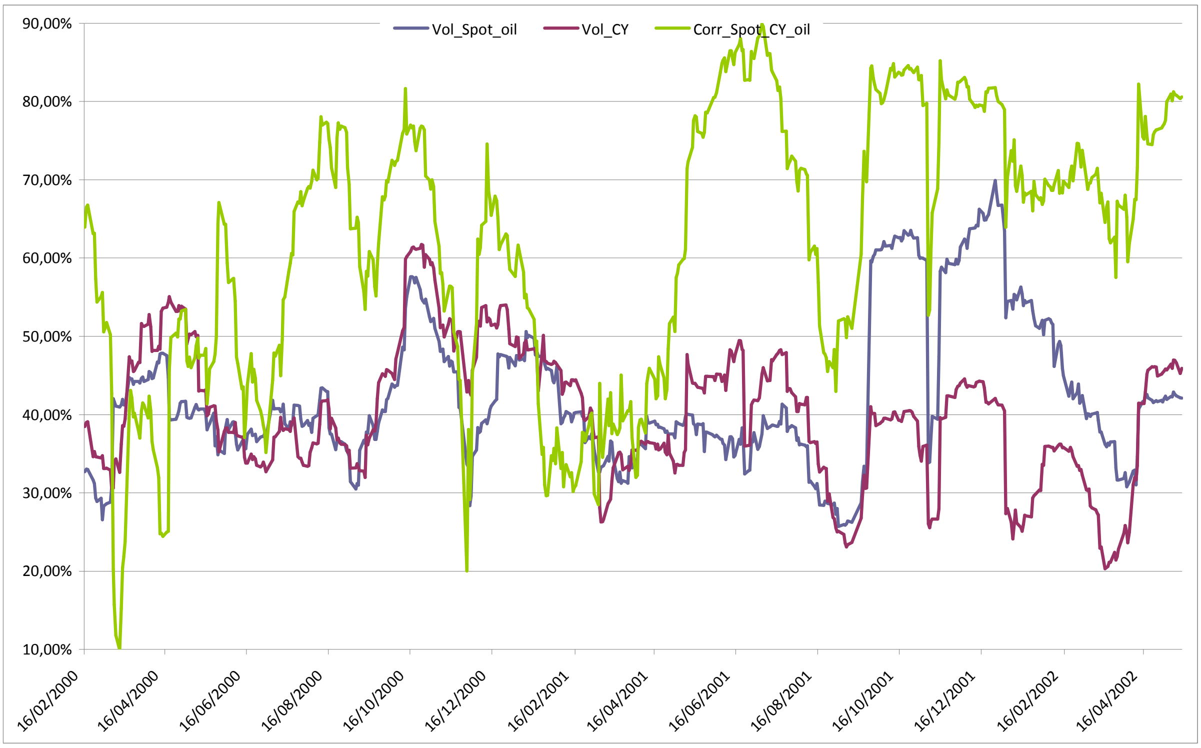

Figure 2. Volatilities and correlation of the oil price and convenience yield from the 16/02/2000 to 16/04/2002. The oil price and the convenience yield are factors of figure 1. The volatilities and the correlation are computed using a daily trading frequency and a window of 60 trading days.

As mentioned earlier in the abstract, the heteroskedasticity of the spot price and the convenience yield is a feature already investigated in the literature about commodity derivatives (Nielsen and Schwartz, 2004; Liu and Tang, 2011). However, these papers model the dynamics of the volatilities of the convenience yield and the spot price as a function of the convenience yield, relying on the tractability of the generalized affine Cox-Ingersoll processes (Dai and Singleton, 2000). A comparison of Figures 1 and 2 suggest nevertheless that the apparent stationary mean-reverting pattern of the volatilities does not stem from a mere scaling of that of the convenience yield. The paper of Richter and Sørensen (2006) do take into account the volatilities of the spot price and the convenience yield as an independent factor. Their modeling is achieved in the spirit of Heston (1993) so that the volatility of the spot price and that of the convenience yield are proportional to one another. Figure 2 demonstrates that this fixed linked between the two volatilities is very unlikely for the oil market.

In addition, in these commodity models, the correlation between the convenience yield and the spot price is either constant (Nielsen and Schwartz, 2004; Richter and Sørensen, 2006) or its dynamics results from the heteroskedasticity of the modeling and, in particular is a function of the convenience yield (Liu and Tang, 2011). This link contradicts Figure 2, which suggests an independent modeling of the correlation for the oil market.

To independently model the correlation and the volatilities of the commodity spot price and the convenience yield, we rely on the affine specification on the cone of semi-definite positive affine process.[2] These processes are now fully specified from a mathematical standpoint (Cuchiero et al. 2011). We denote by $x_t \equiv \log(S_t)$, δ_t the (log) commodity spot price and the commodity convenience yield, respectively. $\sigma_{xt}, \sigma_{\delta t}$, designate the volatilities of the spot price and the convenience yield, respectively. $\rho_{x\delta t}$ stands for the instantaneous correlation between the spot price and the convenience yield. By definition, $\sigma_{x\delta t} \equiv \sigma_{xt}\sigma_{\delta t}\rho_{x\delta t}$ stands for the instantaneous covariance between the spot price and the convenience yield. In this version, for ease of exposition, we directly model the various variables under the risk neutral probability, Q. A discussion on the link between the risk neutral and historical probabilities for the case of affine positive matrix can be found in Barushi et al. (2010).

[2] The reader eager to study affine processes valued in the semi definite positive matrix cone can consult the original paper of Bru (1991) to get the intuition from an historical standpoint. A more practical introduction for financial economists are the various papers of Gourieroux (Gourieroux, 2006; Gourieroux et al. 2007; Gourieroux, and Sufana, 2007a,b). Various financial economists and practionners made these processes popular by studying the modelling of the term structure of interest rates, rainbow options and portfolio management (Buraschi et al. 2007, Buraschi et al. 2010 ; Branger and Muck, 2012; Da Fonseca et al. 2007 ; 2008a,b,c). As already mentioned Cuchiero et al. (2011) provide a complete mathematical framework to study these processes.

To focus on the dynamics of the variance-covariance structure of the spot price and the convenience yield, we assume that the risk free rate, r, is constant. Similarly, we denote by κ and θ the constant speed of mean reversion and long term mean of the convenience yield, respectively. Moreover, " ' " designates the transpose symbol, $M^{\frac{1}{2}}$ stands for the positive square root of a semi definite positive matrix M, Tr denotes the trace of a matrix, and $B_{it}, Z_{it}, B_{ijt}, i, j \in \{x, \delta\}$ designate independent Brownian motions under the probability Q. We are now equipped to display the dynamics of the spot price, the convenience yield as well as their variance-covariance structure (Eqs. 1a,b):

$$dY_t \equiv \begin{bmatrix} dx_t \\ d\delta_t \end{bmatrix} = \begin{bmatrix} r - \delta_t - 1/2\,\sigma_{xt}^2 \\ \kappa(\theta - \delta_t) \end{bmatrix} dt + \Sigma_t^{\frac{1}{2}} \left\{ \begin{bmatrix} dB_{xt} & dB_{x\delta t} \\ dB_{\delta xt} & dB_{\delta t} \end{bmatrix} \begin{bmatrix} \rho_x \\ \rho_\delta \end{bmatrix} + \sqrt{1 - \rho'\rho} \begin{bmatrix} dZ_{xt} \\ dZ_{\delta t} \end{bmatrix} \right\} \quad \text{(1a)}$$

$$d\Sigma_t = \left[\mathrm{K}\left(\bar{\Sigma} - \Sigma_t\right) + \left(\bar{\Sigma} - \Sigma_t\right)\mathrm{K}' \right] dt + \Sigma_t^{\frac{1}{2}} \begin{bmatrix} dB_{xt} & dB_{x\delta t} \\ dB_{\delta xt} & dB_{\delta t} \end{bmatrix} R + R' \begin{bmatrix} dB_{xt} & dB_{x\delta t} \\ dB_{\delta xt} & dB_{\delta t} \end{bmatrix}' \Sigma_t^{\frac{1}{2}}, \quad \text{(1b)}$$

where $\Sigma_t \equiv \begin{bmatrix} \sigma_{xt}^2 & \sigma_{x\delta t} \\ \sigma_{x\delta t} & \sigma_{\delta t}^2 \end{bmatrix}$, $R \equiv \begin{bmatrix} r_x & r_{x\delta} \\ r_{\delta x} & r_\delta \end{bmatrix}$. $\mathrm{K} \equiv \begin{bmatrix} \kappa_x & \kappa_{x\delta} \\ \kappa_{\delta x} & \kappa_\delta \end{bmatrix}$, $\bar{\Sigma} \equiv \begin{bmatrix} \bar{\sigma}_x & \bar{\sigma}_{x\delta} \\ \bar{\sigma}_{x\delta} & \bar{\sigma}_\delta \end{bmatrix}$.

The dynamics displayed by Eqs (1a,b) deserve the following explanations. $\rho' \equiv [\rho_x \quad \rho_\delta]$, $\rho'\rho \leq 1$, characterizes the correlation between on the one hand, the dynamics of the spot price and the convenience yield, and, on the other hand, the dynamics of their variance-covariance structure, Σ_t. R is a matrix, which links the matrix covariance of the spot price and the convenience yield, Σ_t, to its own variance covariance structure. K is a (semi) definite positive matrix that stands for the speed of mean reversion of the variance-covariance process (Branger and Muck, 2012). $\bar{\Sigma}$ is a (semi) definite positive symmetric matrix (Curichio et al., 2011; Branger and Muck, 2012). The following condition must hold to ensure the well posedness of Eq (1b) (Curichio et al., 2011):[3]

$$\mathrm{K}\bar{\Sigma} + \bar{\Sigma}\mathrm{K}' \succ RR', \quad \text{(2)}$$

[3] More general drift structure of Eq (1b) holds (Curichio et al., 2011). For economic practicability in this first investigation, we believe that our presentation is thorough enough.

where $\succ$ designates the partial order induced by semi definite positiveness on the set of semi definite positive symmetric matrix.

We designate by $F_t(\tau)$, the time-t futures price of time to maturity $\tau \equiv T - t$. The futures price is a martingale under the risk neutral probability (Duffie and Stanton, 1992):

$$F_t(\tau) = E_t\left[e^{x_T}\right], \tag{3}$$

where $E_t \equiv E\left[\left|F_t\right.\right]$ designates expectation conditional on time-t information, $F_t \equiv \sigma(Y_u, \Sigma_u, u \leq t)$, under the risk neutral probability. As shown by Cuchiero et al. (2011), the conditional characteristic function of Eqs (1a,b) is affine. As a consequence, by analytic extension or by simply applying the Feynman-Kac formula to Eq. (3), we can show that the futures price is of the exponential affine type as follows:

$$\log(F_t(\tau)) = f(\tau) + F_Y(\tau)' Y_t + Tr[F_\Sigma(\tau)\Sigma_t], \tag{4}$$

where $f(\tau), F_Y(\tau), F_\Sigma(\tau)$ are respectively real, vector and matrix valued functions of time to maturity. These functions are available from the author upon request.

In order to assess the parameters of the dynamics (1a,b) via the measurement equation (4), we use data from oil futures prices traded on the Nymex from the 02/12/1996 to the 31/05/2004. We display four times to maturities: 1, 3, 6 and 11 months. Together, these contracts represent a panel data of 4x1959 points. Their descriptive statistic is sum up in Table 1 and Figure 3.

Table 1. Empirical moments of futures prices and their time to maturities

Contract 1 month		**Contract 3 months**		**Contract 6 months**		**Contract 11 months**	
Price \$ Average (Std. Dev.)	Mat. days Average (Std. Dev.)	Price \$ Average (Std. Dev.)	Mat. days Average (Std. Dev.)	Price \$ Average (Std. Dev.)	Mat. days Average (Std. Dev.)	Price \$ Average (Std. Dev.)	Mat. days Average (Std. Dev.)
22.98 (6.22)	29.74 (8.80)	22.62 (5.75)	90.56 (8.83)	22.03 (5.15)	181.92 (8.85)	21.22 (4.40)	334.12 (8.83)

Table 1. First, average, and second, standard deviation, empirical moments of the four futures prices and their time to maturity for the sample covering the period from 02/12/1996 to 31/05/2004. Mat. stands for time to maturity and Std. Dev. for standard deviation.

Table 1 displays the first two empirical moments of the various futures price, i.e., the average price and the standard deviation. These two empirical moments display two well-known properties of the oil market: i) the oil market is on average in backwardation: the average futures price is a decreasing function of time to maturity; ii) the oil market exhibits the so called Samuelson effect: the standard deviation of the futures price is also a decreasing function of the time to maturity.

In addition, Table 1 gives the average time to maturity of the futures contracts as well as the standard deviation of these times to maturity. Indeed, the time to maturity of a futures contract decreases as time goes by and new futures contracts are only created once a month. As a consequence, the design of futures prices with exactly constant time to maturity can not be achieved (Schwartz, 1997).

Figure 3. Empirical term structure of volatilities as a function of time from 24/02/1997 to 31/05/2004

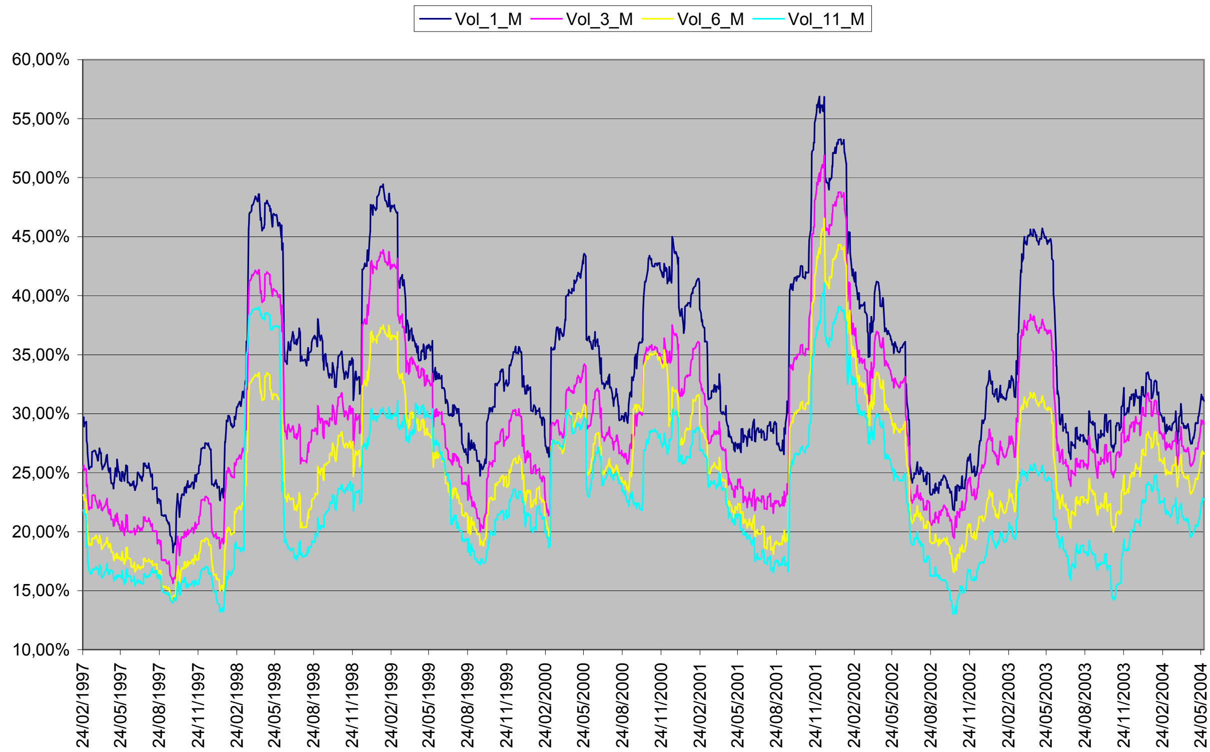

Figure 3. Empirical volatilities of the futures prices for the contracts with time to maturities 1, 3, 6 and 11 months from 24/02/1997 to 31/05/2004. The volatilities are computed using a daily trading frequency and a window of 60 trading days.

Figure 3 displays the empirical volatilities of the four futures contracts as a function of time from 24/02/1997 to 31/05/2004. As a consequence, for a given date t, the vertical analysis of Figure 3 provides the term structure of volatility. First, Figure 3 completes Table 1 and shows that the Samuelson effect holds at any date *t*: the four curves never intersect each other. Second, Figure 3 demonstrates an additional feature of the data that should be handled by the dynamics described by Eqs (1a,b): the Samuelson effect seems all the more pronounced than the level of the volatilities is high. This stylized fact is in line with the theory of storage because "stockouts" should have impact mainly on nearby contracts.

Several possibilities to carry out an empirical analysis of our model arise. One of them is to use the maximum likelihood function relying on the transition function as outlined by Gourieroux (2006). Another possibility would be to rely on the tractability of the characteristic function of Eqs (1a,b) and to perform a Generalized Methods of Moments (GMM) in the frequency space as shown by Da Fonseca et al. (2008b) – see also Carrasco and Florens (2000) for the exposition of the Generalized methods used in Da Fonseca et al. (2008b). However, these methods would require the specification of our model under the historical probability: we leave this case for further investigations.

As a first approach, we choose to perform a simple calibration to get the risk-neutral parameters of Eqs (1a,b) through the measurement Eq. (4). In addition, because we are mainly interested in the dynamics of the variance-covariance structure of the spot price and the convenience yield, we choose to calibrate the volatilities of the term structure of futures price as displayed by Figure 3. Indeed, using the affine property of the generator of Eqs (1a,b), see Cuchiero et al. (2011), and Eq. (4), we can show that the conditional instantaneous variance term structure of futures price is an affine function of its variance-covariance matrix. As a consequence, a simple two steps procedure, as described in Cortazar and Schwartz (2003), can be used to calibrate the dynamics of the variance-covariance Σ_t.

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