Default Greeks under an objective probability measure*

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Abstract

Correct estimation of default probabilities are becoming an increasingly important element of financial institution's measurement and management of credit exposures. Growing stress on proper rating systems driven by regulators also motivates for a better understanding of these issues. This article focus on the use of option based models in credit risk management. We present a framework to derive an objective probability of default directly, and its comparative statics (or socalled "Greeks") based on the BlackScholesMerton model. We then demonstrate the use of these concepts with several numerical examples. In special, we elucide the difference between risk neutral and objective (or real) probabilities of default and its implication for credit risk management and capital allocation strategies. Effort should though be put on the transformation of the risk measures as wrong estimates of expected default frequencies could lead to a situation where to little capital is allocated to less risky project and as a consequence destroy shareholder value.

JEL classification: G13, G21, G33

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1 Introduction

The credit related business have become progressively more diverse and complex and the number of counterparts has grown rapidly, straining the limits of traditional methods of controlling and managing credit risk. As a consequence, many banks and large financial institutions have developed models to assess credit risk and to allocate economic capital to different segments of their portfolios (see Altman et al. (1998) and Saunders (1999) for an overview of these tools). The results are more structured or formal systems for approving loans, portfolio monitoring and management reporting, capital allocation, risk adjusted performance measurement and loan pricing.[1]

In any credit risk management system, a correct estimation of the probability of default of a counterpart is crucial. Banks use estimated default probabilities (EDF) in computing expected and unexpected loss, economic capital and risk adjusted profitability measures and pricing guidelines. An *overestimation* of EDF will lead to an overestimation of expected and unexpected loss and economic capital and an underestimation of risk adjusted performance. This could lead to less capital allocated than ideal to these counterparts and destruction of shareholders value creation. An *underestimation* of of EDF will lead to an underestimation of expected and unexpected loss and economic capital and an overestimation of risk adjusted performance. This could lead to more capital allocated than ideal to these counterparts and again a destruction of shareholders value creation in accordance to the target return ratio. Achievement of getting the right EDF for various corporate clients, should indeed be a concern for most banks.

During the last years, it has emerged several frameworks to analyze credit risk in a consistent manner. (see Crouhy et al. (2000) for an overview of CreditMetrix, KMV, Credit+ and CreditPortfolioView which have become "market standard" models). One approach that has been very popular is the KMV model, probably because of its very strong theoretical underpinnings. The approach is often called the option pricing or structural approach which is based on Black & Scholes (1973) and Merton (1974), hereafter the BSM approach. In this model the default process is endogenous, and relates to the capital structure of the firm. KMV Corporation, a firm specialized in credit risk analysis, has over the last years developed a credit risk methodology, as well as an extensive database based on the BSM model. In KMV's model one assess default probabilities and loss distribution related to both default and migration risks. KMV relies on EDF for each issuer, rather upon the average historical transition frequencies produced by the rating agencies, for each credit class. KMV though uses simplifying assumptions about the transition from risk neutral to "real" EDF to facilitate its implementation (see section 2).

In this paper we discuss how damaging in practice these assumptions could be to satisfactory capture the actual complexity of credit risk and performance measurement. More precisely, we aim to rewrite the BSM model to explicitly state the objective (as opposed to risk neutral) probability of default as a function of the other variables in the model, and calculate its comparative statics

[1] A strong incentive to develop these models has been accelerated by the release of a consulting paper and later revisions by the Basel commitee on banking supervision (see BIS (1999,2001,2003)). There is an ongoing process for modification and enhancement of the BIS proposal (see e.g. Altman (2001) where capital adequacy and ratings are debated).

From there we will discuss numerical examples on the difference between risk neutral and objective probability of default and the consequences for credit risk management. In spite of the popularity of the BSM model and its elaboration into the KMV model, as far as we know such analysis has not been performed. EDF based on risk neutral default probabilities could give quite misleading information when calculating Credit VAR and RAROC. If the mapping process for the probability measures is not done properly, we might reject lending prospects that should have been accepted and/or accepting counterparts that should have been rejected according to a wrongly calculated RAROC figures. Our paper clearly show this effect through comprehensive numerical examples of a modified BSM framework.

Sensitivity analysis of the EDF is an important element in monitoring credit risk. This provides the credit analyst with the possibility to perform what-if analyses regarding the model's various inputs such as changes in the equity valuation, capital structure and business risk of the firm. It is also important to give the decision maker information on when the sensitivities measures are at their highest levels as well as interaction effects among the sensitivity measures. Through a comparative static analysis and various numerical experiments, we demonstrate these effects. We suggest to use these sensitivity measures for the default probability as an objective examination and use them in combination with other sources of information when making credit risk decisions.

The remainder of this paper is organized as follows. Section 2 briefly discusses the BSM/KMV model and the use of option based models in credit risk management. In Section 3 we derive a framework for an objective probability of default, as well as comparative statics or default greeks for our default probability measure. In Section 4 we perform several numerical examples and discuss the difference between risk neutral and objective default probabilities and its implications. Section 5 concludes the paper and gives some guidelines for further work

2 Option based default modelling

2.1 The BSM approach to default prediction

Thirty years ago, Merton (1974) introduced a contingent claims approach to valuing corporate capital categories using the Black and Scholes (1973) and Merton (1973) option pricing theory. The approach is based on the fact that the combination of limited liability and a leveraged capital structure creates option-like payoff structures, in which the values of a firm's debt and equity are contingent on the market value of its assets. Thus, the value of equity can be modelled as a call option on the firm's assets, with the face value of debt as exercise price and debt maturity as the option's time to maturity. In a similar way, the value of debt can be modelled as a put option.This approach has served the academic profession well in providing an internally consistent and reasonably intuitive framework for valuing corporate debt and equity.[2]

[2] The BSM approach is what we call the first "structural" model of default, in which credit events are triggered by movements of the firm's value relative to some threshold or barrier. Examples of structural models after BSM are: Black & Cox (1976), Geske (1977), Brennan and Schwartz (1977,1978,1980), Leland (1994), Longstaff & Schwartz (1995), Leland & Toft (1996), Zhou (1997), Anderson and Sundaresan (1996, 2000), Ericsson and Reneby (1998)

In the BSM framework we can derive closed form formulas for the value of equity and debt, the default spread of the bond, the default probability and the expected recovery value in case of default, the expected discounted shortfall and the overall cost of default. (see Crouhy and Galai (1997) and Cossin (2001) for an excellent treatment of these issues).While each of these items are critical to the management of credit portfolios, we believe none is more important than the probability of default.

The BSM model starts out with an assumption regarding the stochastic process the firm value follows. In the BSM model, the firms asset value V_t is assumed to follow a standard geometric Brownian motion, i.e:

$$V_t = V_0 e^{(\mu_V - \frac{\sigma^2}{2})t + \sigma_V \sqrt{t} Z_t} \tag{1}$$

with $Z_t \sim N(0,1)$, μ_V and σ_V^2 being respectively the mean and variance of the instantaneous rate of return on the asset of the firm. V_t is lognormally distributed with expected value at time t, $E(t) = V_0 e^{\mu_V t_t}$.It is further assumed that the firm has a very simple capital structure, as it is financed only with equity, E_t, and a single zero-coupon debt instrument maturing at time T, with face value F, and current market value D_t. The firm's balance sheet can be represented as in Table 1.

Table 1

Balance sheet of a firm in the BSM model

Assets		Liabilities	
Risky assets	V_t	Debt	D_t
		Equity	E_t
Total	$\overline{V_t}$		$\overline{V_t}$

In this framework, default only occurs at maturity of the debt obligation, when the value of assets is less than the promised payment, F, to the bondholders. Figure 1 shows the distribution of the asset's value at time T, the maturity of the zero-coupon debt, and the probability of default which is the shaded area below F.

The probability of default is given by

$$P_{Default} = \Pr[V_t \leq F] \tag{2}$$

The BSM model assumes that the normalized log-returns over any period of time is normally distributed with mean 0 and variance 1. If we translate the

and Ericsson (2000).

Another competing class of models are the "reduced form" models where credit events are specified in terms of some exogenously defined process. Examples of such models are Jarrow and Turnbull (1995), Jarrow et. al. (1997), Duffie & Singleton (1998), Lando (1998), and Lando (2000).

An overview of both structural and reduced form models can be found in Bohn (2000), Cossin and Pirotte (2001), Duffie and Singleton (2003) and Schonbucher (2003).

probability of default in (2) into a normalized threshold, we get;

$$P_{Default} = \Pr\left[\frac{\ln\frac{F}{V_0} - (\mu_V - \frac{\sigma_V^2}{2})t}{\sigma_V\sqrt{t}} \geqq Z_t\right] \tag{3}$$

or

$$P_{Default} = \Pr\left[Z_t \leq -\frac{\ln\frac{V_0}{F} + (\mu_V - \frac{\sigma_V^2}{2})t}{\sigma_V\sqrt{t}}\right] \equiv N(-d_2) \tag{4}$$

where the normalized return $\frac{\ln\frac{V_0}{F}+(\mu_V-\frac{\sigma_V^2}{2})t}{\sigma_V\sqrt{t}}$is $N(0,1)$.Z_t is simply the threshold point in the standard normal distribution corresponding to a cumulative probability of $P_{Default}$. Then the critical asset value F which trigger default is such that $Z_t \leq -d_2$ where $d_2 \equiv \frac{\ln\frac{V_0}{F}+(\mu_V-\frac{\sigma_V^2}{2})t}{\sigma_V\sqrt{t}}$. This is called the "distance to default" in the KMV model (see KMV (2001)). Note that d_2 is different from its equivalent in the Black and Scholes formula since, here we work with the "actual" or objective probability of default instead of the "risk neutral" return distributions, so that the drift term in d_2 is the expected return on the firm's assets, instead of the risk-free interest rate as in Black and Scholes (1973).

2.2 Usage of BSM model in credit risk management - The KMV approach

Historically, banks have ignored stock market prices in their lending decisions although its importance for default predictions has been stressed for many years (e.g. see Altman (1968) and Altman et.al. (1977)). The BSM approach to bankruptcy prediction bring market capitalization into the lending equation and should in this context enhance credit decisions made by banks and financial institutions. By being based on stock market data rather than "historic" book value accounting data, such an approach is forward looking, and it has a strong theoretical underpinnings from financial theory where equity is viewed as a call option on the assets of the firm.

In recent years KMV has created a commercial application based on the BSM framework[3] Their application, the Credit Monitor, produces and updates default predictions for all major companies and banks that have equity publicly traded[4]. The calculations of EDF in KMV's model is based on 3 steps. In the first step, the market value and volatility of the firm is estimated from the market value of its stock, the volatility of its stock, and the book value of its liabilities. In the second step, the firms distance to default is calculated. Distance to Default (DD) is a measure is constructed that represent the number of standard deviations from the expected firm value to the default point.Finally, an *empirical mapping* is determined between the distance to default and the default rate, based on the historical default experience of companies with different distance

[3]See KMV (2001), KMV (2002). Altman et.al. (1998) and Saunders (1999) and Crouhy et al. (2000) also gives an extensive description of the KMV framework used in credit risk management.

[4]KMV has now extended their model into credit portfolio modeling. The have also created a "private firm" model for non-traded companies.

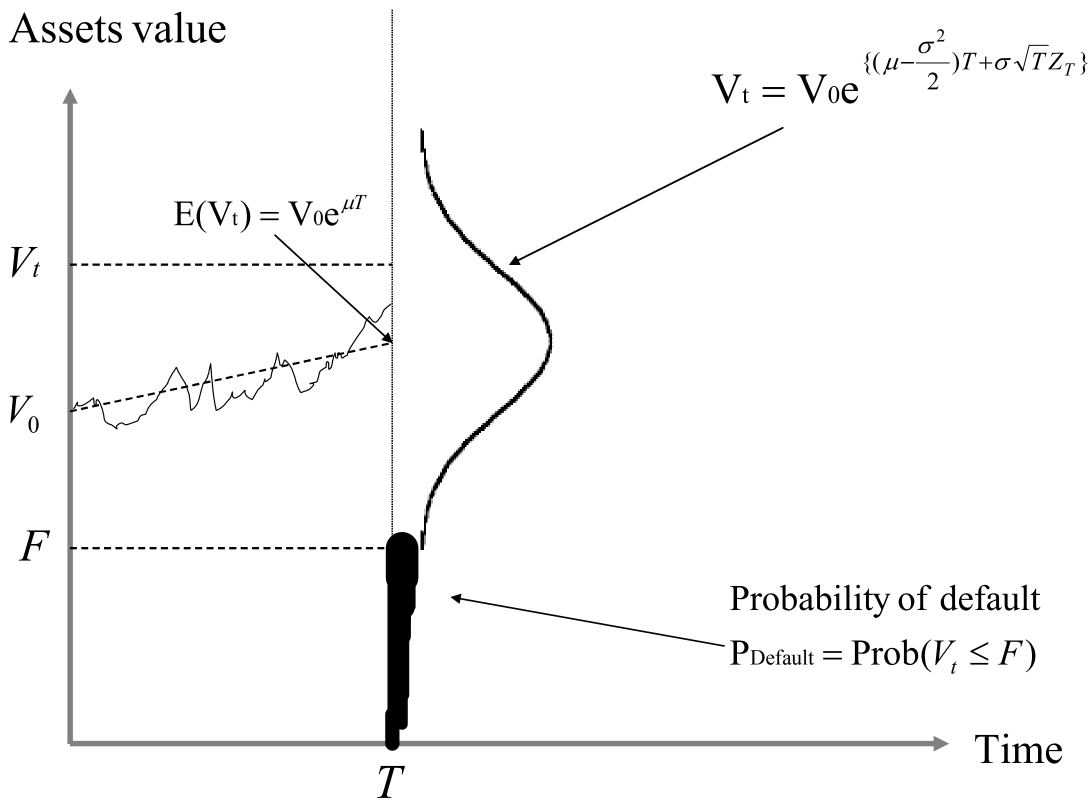

Figure 1: Distribution of the firm's assets value at maturity of the debt obligation

to default values[5]. KMV calculate the risk adjusted default probability by comparing the *calculated distance to default* and the *observed actual default rate* for the same group of firms. A smooth curve fitted to those data yields the EDF as a function of the distance from default.Given the current capital structure of the firm, i.e. the composition of its liabilities: equity, short-term and long-term debt, convertible bonds, etc. once the stochastic process for the asset value has been specified, then the estimated probability of default (EDF) for any horizon can be derived. Fig. 1 in the previous section depicts how the probability of default relates to the distribution of asset returns and the capital structure of the firm, in the simple case where the firm is financed by equity and a zero-coupon bond.

The actual uses to which financial institutions (mostly commercial banks) put the KMV model vary widely (see Altman et. al. (1998)). At one end of the spectrum, some banks use a company's EDF as one more piece of information among the general sources it consults. Other institutions have formally added EDF values as supplemental information to the traditional credit analysis. Still others use the EDF values as a way of assigning internal risk ratings to their clients. Some use EDF as an early warning tool in the portfolio review group to alert loan officers about changes in a company's risk profile, and more and more institutions uses EDF's as the sole indicator of credit risk in loan pricing as well as in risk adjusted valuation related to the trading of bank debt.

The issue we want to discuss in this paper is the mapping of distance to default (DD)[6] to the estimated probability of default (EDF) for given time horizons. Based on historical information on a large sample of firms, which includes firms which defaulted one can estimate, for each time horizon, the proportion of firms of a given ranking, say DD=3, which actually defaulted after 1 year. This proportion, say 70BP or 0.7%, is the EDF (see figure 2).

2.3 Risk neutral versus objective probability of default

KMV's pricing model is based upon the "risk neutral" valuation model, also known as the Martingale approach to the pricing of securities, which derives prices as the discounted expected value of future cash flows. The expectation is calculated using the so-called risk neutral probabilities and not the actual probabilities as they can be observed in the market place from historical data or EDF's. When the objectives are to assess the credit risk of various positions and not price contingent claims, the objective or "actual" default probability has to be determined.

The differences between actual and risk-neutral default probabilities reflect the risk premium required by market participants to take on the risk associated with default. In general, default-risk premium reflect aversion to both the risk

[5] In the case of private companies (see KMV (2001)) for which stock price and default data are generally unavailable, KMV uses essentially the same approach by estimating the value and volatility of the private firm directly from its observed characteristics and accounting data. These estimates, however, are based on public company data.

[6] In the KMV model the "distance to default" (DD) is modified from the theoretical definition we made earlier. DD is the number of standard deviations between the mean of the distribution of the asset value and a critical thereshould or "default point". In KMV this default point is set at the par value of current liabilities including short term debt to be serviced over the time horizon, plus half the long term debt.

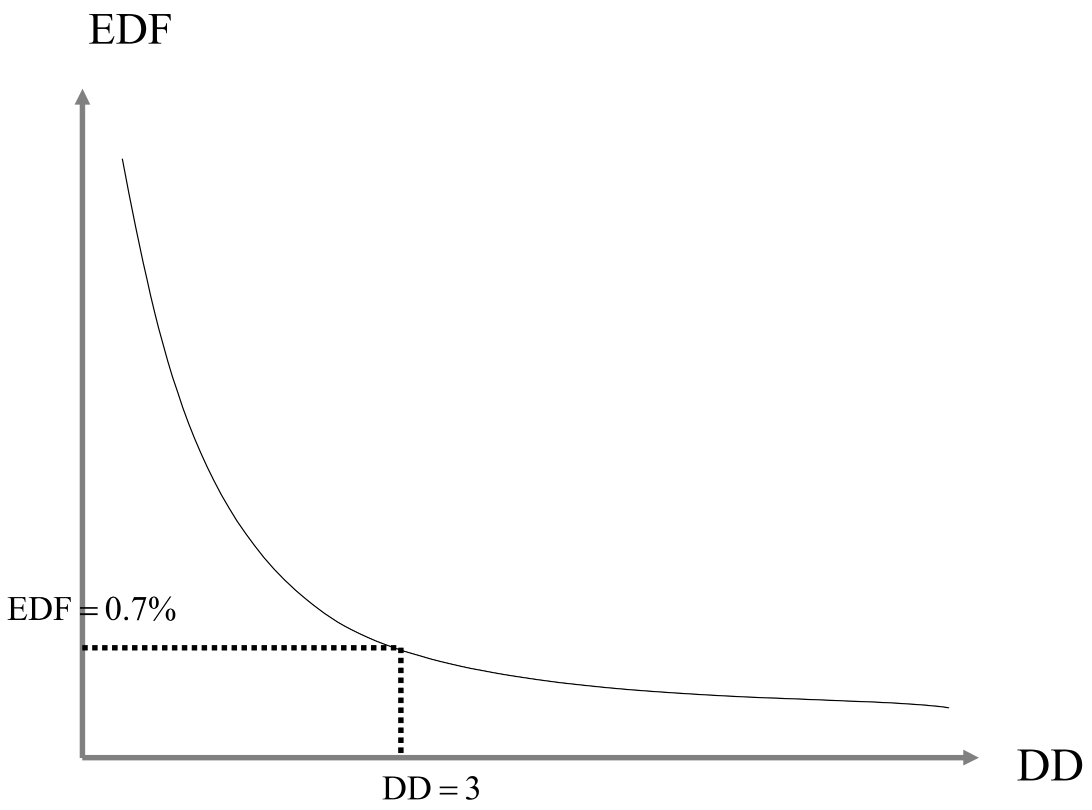

Figure 2: Mapping of the "distance-to-default" (DD) into the "expected default frequency" (EDF)

of timing of default, and to the risk of severity of loss in the event of default. Since the objective and risk neutral distributions of the firm's value have the same variance (the diffusion terms are the same), and the objective distribution must generally have a mean greater than the risk free rate (the drift is generally higher), it follows that the risk neutral distribution implies a higher default probability. (see Deliandes and Geske (2003), Duffie and Singleton (2003) for a more detailed discussion on these issue).

If EDF *overstates* the default rate for a typical obligor, it can lead to adverse selection of corporate customers in banks. Indeed, if the pricing of loans is based on a EDF that is to high, then a typical customer will be overcharged and may have incentives to *leave*, while the worst obligors will benefit from an advantageous pricing with regard to their actual risk. Therefore, in using risk neutral default probabilities as a basis for real default probabilities, one should be very concerned of the implications it induces.

3 Deriving objective default probabilities and their sensitivities

3.1 Introduction

Following last chapters motivation, we need some sort of objective default probabilities when we are to implement subjective trading strategies, measure credit risk on portfolios or perform credit allocation among business units. Since "real" default probabilities are not directly observable, we need to know the mapping between actual and risk neutral default probabilities. This section describe a mapping that can be used by practitioners that connect risk neutral and actual probabilities. We also describe a procedure on how to derive comparative statics from this measure.

The problem becomes evident from equation (4) In order to calculate the actual default probability, we need to determine the following variables:

1. The value of assets, V
2. The volatility of asset value, σ_V
3. The drift of asset value, μ_V

Our suggestion is to use a two step procedure, first deriving the value of the firm and its volatility[7] and subsequently derive the drift of the asset value. Then, having found the objective probability of default, we derive the comparative statics of this measure w.r.t. the value and the volatility of the asset, the drift of the asset and the time horizon of debt.

Under the risk neutral probability measure, the expected return on all securities (here: equity and debt) is the risk free rate of return, r, for any horizon T. Therefore, the risk neutral probability of default, Q, is defined as the probability of default such that the value of the assets at T falls below the default point, F , under the modified risk neutral process for the asset value V_t^*.The risk neutral probability of default is given by

[7] See also KMV (2001) and Crouhy (2000) for a description of this procedure.

$$Q = P^*_{Default} = \overset{*}{\Pr}[V_t^* \leq F] \quad (5)$$

$$= \overset{*}{\Pr}\left[\frac{\ln\frac{F}{V_0} - (r - \frac{\sigma_V^2}{2})t}{\sigma_V\sqrt{t}} \geqq Z_t\right] \quad (6)$$

or

$$= \overset{*}{\Pr}\left[Z_t \leq -\frac{\ln\frac{V_0}{F} + (r - \frac{\sigma_V^2}{2})t}{\sigma_V\sqrt{t}}\right] \equiv N(-d_2^*) \quad (7)$$

Which is similar to (4) apart from the asset drift μ_V is now replaced with the risk free rate of return r[8].

The well known BSM formula where equity is viewed as a call option on the firms asset ($max(V-F,0)$ at T),.is given by

$$E = VN(d_1^*) - Fe^{-rT}N(d_2^*), \text{ with} \quad (8)$$

$$d_1^* = \frac{\ln(V/F) + (r + \sigma_V^2/2)T}{\sigma_V\sqrt{T}} \text{ and} \quad (9)$$

$$d_2^* = \frac{\ln(V/F) + (r - \sigma_V^2/2)T}{\sigma_V\sqrt{T}} \quad (10)$$

We can in this approach also obtain analytical expressions for the value of debt, the yield on the debt, the default probability and the expected recovery value in case of default (see Crouhy and Galai (1997) and Cossin (2001) for a discussion of these issues). While each of these items is critical to the management of credit portfolios, the scope of this paper is to analyze the default probability.

3.2 Finding asset value and asset volatility

If all corporate claims (equity and debt classes) was publicly traded, the value of the firm would be observable. The firms volatility could in turn have been estimated by using a time series of asset values. This is not the usual case (only parts of equity and debt are traded in public). We must then use an alternative approach in estimating V and σ_V.

The method used by KMV (see KMV(2001) and Crouchy(2000) for a description) is to derive the asset value and asset volatility from the BSM and an additional identity concerning the relationship between the asset value/asset volatility and the equity value/equity volatility. We exploit the option nature of equity to derive the market value and volatility of the firm's underlying assets implied by the market value of the equity. In particular, we solve backwards

[8] In the risk neutral world the asset follow the process $\frac{dV_t}{V_t} = rdt + \sigma_V dZ_t$ where Z_t is a standard Brownian motion, and $Z_t - Z_0$ is normally distributed with zero mean and variance equal to T. In the "real" world the asset follow the process $\frac{dV_t}{V_t} = \mu_V dt + \sigma_V dZ_t$. Note that the term $\sigma_V dZ_t$ is equal for these processes. That is, only the drift terms are different.

from the option price and option price volatility for the implied asset value and asset volatility.

We know that, according to our assumptions, the processes for the assets (V), equity (E) and debt (D) can be represented by the following stochastic differential equations;

$$dV_t = \mu_V V_t dt + \sigma_V V_t dZ_t \tag{11}$$

$$dE_t = \mu_E E_t dt + \sigma_E E_t dZ_t \tag{12}$$

$$dD_t = \mu_D D_t dt + \sigma_D D_t dZ_t \tag{13}$$

Here V_t, E_t and D_t are the values of assets, equity and debt respectively[9], μ_V, μ_E and μ_D represent the drift in the values of assets, equity and debt. $\sigma_V, \sigma_E, \sigma_D$ are the volatilities of asset, equity and debt. At any time, the following definition relations obtain: $V_t = E_t + D_t$ and $dV_t = dE_t + dD_t$, that is, the value of the firm's asset must be equal to the value of debt and equity and the change in the value of the assets must be equal to the change in the values of equity and debt.

By Ito's lemma, we can also represent the processes for the equity and debt as:

$$dE_t = \left(\frac{\partial E}{\partial t} + \mu_V V_t \frac{\partial E}{dV} + \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 E}{\partial V^2}\right) dt + \sigma_V V_t \frac{\partial E}{\partial V} dZ_t \tag{14}$$

$$dD_t = \left(\frac{\partial D}{\partial t} + \mu_V V_t \frac{dD}{dV} + \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 D}{\partial V^2}\right) dt + \sigma_V V_t \frac{\partial D}{\partial V} dZ_t \tag{15}$$

By comparing the diffusion terms in the equity process in (12) and (14) we get the following[10] relationship:

$$\sigma_E E_t = \sigma_V V_t \frac{\partial E}{\partial V} = \sigma_V V_t N(d_1^*) \tag{16}$$

In which $N(d_1)$ is the hedge ratio or the "delta" in standard option terminology. Substituting equation (9) and (10) into (8), substituting (9) into (16) and reorganizing terms we can form the following system of 2 nonlinear equations:

$$VN\left(\frac{\ln(V/F) + (r + \sigma_V^2/2)T}{\sigma_V \sqrt{T}}\right) - Fe^{-rT} N\left(\frac{\ln(V/F) + (r - \sigma_V^2/2)T}{\sigma_V \sqrt{T}}\right) - E = 0 \tag{17}$$

$$\sigma_E E - \sigma_V VN\left(\frac{\ln(V/F) + (r + \sigma_V^2/2)T}{\sigma_V \sqrt{T}}\right) = 0 \tag{18}$$

Note that V,and σ_V,are the unknown variables in (17) and (18), the remaining parameters E, σ_E, r, F and T are assumed to be found or estimated outside the model. This system of 2 simultaneous nonlinear equations and 2 unknown variables can be solved using a standard Newton Raphson algorithm[11].

[9] We should have quoted E_t and D_t as E(V,t) and D(V,t), but for keeping the notation simple this is dropped.

[10] This is referred to as the proposition of unique decomposition (see e.g. Bjørk (1998) and Neftci (2000)).

[11] In Appendix C we have elaborated an numerical example using Maple.

The calculation above must be taken carefully. The model linking equity and asset volatility given by (16) only holds instantaneously. If the market value of equity moves far from the starting value, then (16) gives biased results (see Crouhy (1997) and KMV(2001))).

3.3 Finding the drift of assets

Having found the asset value V and the volatility of the asset value σ_V, the next step is finding the drift of the asset μ_V. This drift can be done by using equations (12) and (14). We define the following measures (see appendix B for more details):

$$\text{EquityDelta } \Delta^E = \frac{\partial E}{\partial V} = N(d_1^*) > 0 \tag{19}$$

$$\text{EquityGamma } \Gamma^E = \frac{\partial^2 E}{\partial V^2} = \frac{n(d_1^*)}{V\sigma\sqrt{T}} > 0 \tag{20}$$

$$\text{EquityTheta } \Theta^E = \frac{\partial E}{\partial t} = -\frac{Vn(d_1^*)\sigma}{2\sqrt{T}} - rFe^{-rT}N(d_2^*) \quad \lesseqgtr 0 \tag{21}$$

These measures are similar to the well known standard sensitivity measures (option Greeks) for a European call option. Having found expressions for Δ^E, Γ^E, and Θ^E, we can now compare the drift terms of equations (12) and (14) and solve for drift of the assets μ_V:

$$\mu_E E = \frac{\partial E}{\partial t} + \mu_V V \frac{\partial E}{\partial V} + \frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 E}{\partial V^2} \tag{22}$$

$$\mu_E E = \Theta^E + \mu_V V \Delta^E + \frac{1}{2}\sigma_V^2 V \Gamma^E \tag{23}$$

$$\Longrightarrow \mu_V = \frac{\mu_E E - \Theta^E - \frac{1}{2}\sigma_V^2 V^2 \Gamma^E}{V\Delta^E} \tag{24}$$

In the approach followed here we assume that the drift (growth rate) of equity μ_E, can be found or estimated from the stock market. In order to estimate μ_E, we need a pricing model such as the CAPM. According to CAPM we have:

$$\mu_E - r = \beta\pi \tag{25}$$

Here β is the beta of the equity with the market $\beta = \frac{cov(R_E,R_M)}{var(R_M)} = \rho\frac{\sigma_E}{\sigma_M}$ where R_E and R_M denote the return on the firm's equity and the market return, respectively. σ_E, σ_M andρ is the volatility of the firms equity, the volatility of the market portfolio and the correlation between the equity return and the market return respectively.

π is the market risk premium for a unit of beta risk or the "market price of risk". It is given by;

$$\mu_M - r = \pi \tag{26}$$

where μ_M denote the expected return on the market portfolio. It follows that:

$$\frac{\mu_E - r}{\sigma_E} = \frac{\beta\pi}{\sigma_E} = \rho\frac{\pi}{\sigma_M} = \rho * S \tag{27}$$

where S indicates the Sharpe ratio, i.e. the excess return per unit of market volatility for the market portfolio.The parameters needed to determine μ_E can be difficult to estimate, especially the market price of risk .π since it could vary over time.

The framework above represent a theoretical sound way of getting the objective probability of default. By using option theory and the CAPM, we can find the asset value and its volatility and drift and and calculate (4) in a consistant way.

3.4 Finding the default probability greeks

With the popularity of credit risk management tools based on the BSM approach, it is important to get a deeper understanding w.r.t. the dynamics of the default probability and how it is affected by the underlying parameters in the model. A sensitivity analysis provides the credit analyst with the possibility to perform what-if analyses regarding the model's various inputs. These sensitivities, or default Greeks, can be calculated in a straightforward manner as the derivatives of (4) w.r.t. its parameters (V, σ_V, μ_V, T) (see Appendix B):

- The sensitivity of the objective default probability p with respect to a change in the value of the firm V is given by:

$$\frac{\partial p}{\partial V} = \frac{\partial N(-d_2)}{\partial V} = -n(d_2) * \frac{1}{V\sigma_V\sqrt{T}} < 0 \tag{28}$$

The probability of default unambiguously decreases with the firm value. Higher asset values makes it less likely to hit the default barrier F for any asset volatility, asset drift and time horizon.

- The sensitivity of the objective default probability p with respect to a change in the volatility of the firm σ_V is given by:

$$\frac{\partial p}{\partial \sigma_V} = n(d_2) * \left[\frac{\ln(\frac{V}{F}) + T\mu_V + \frac{1}{2}T\sigma_V^2}{\sigma_V^2 \sqrt{T}} \right] > 0 \tag{29}$$

Intuitively, default probability is positively related to the volatility of the firms asset, since more volatile firm are more likely to move close to the default boundary.

- The sensitivity of the objective default probability p with respect to a change in the drift of the firm μ_V is given by:

$$\frac{\partial p}{\partial \mu_V} = n(d_2) * \frac{\partial \left[-\left(\frac{\ln(\frac{V}{F}) + (\mu_V - \frac{\sigma_V^2}{2})T}{\sigma_V \sqrt{T}} \right) \right]}{\partial \mu_V} = -n(d_2) * \frac{\sqrt{T}}{\sigma_V} < 0 \tag{30}$$

Structural default probabilities are unambiguously decreasing in the expected return/drift of asset. Increases in the expected return on asset, increases the drift away from the default barrier and concomitantly decrease the probability of default.

- The sensitivity of the objective default probability p with respect to a change in the time to maturity T is given by:

$$\frac{\partial p}{\partial T} = \frac{\partial N(-d_2)}{\partial T} = n(d_2) * \left[\frac{\ln(\frac{V}{F})}{2\sigma_V T^{3/2}} - \frac{\mu_V}{2\sigma_V \sqrt{T}} + \frac{\sigma_V}{4\sqrt{T}} \right] \gtrless 0 \tag{31}$$

The default probability increases with the time to maturity for firms with low leverage, and rapidly increases and then decreases for highly levered firms. Economically, firms with little debt are more likely to default the larger the time available for "bad" realizations; firms that are very close to default either default quickly, or the fact that they continue to survive is evidence that the value of the firm has moved further away from default. Structural models all have zero probability of default over the next instant, since out of the money options (in this case the "option" is the equity of the firm) are worthless immediately prior to expiry.

Note that the sensitivity measures are the same in a risk adjusted and a risk neutral world apart from the drift term: the risk free rate takes the place of the assets drift rate in the risk neutral world.

To sum up, the results presented in (28), (29) and (30) show that the default probability is a decreasing function of the asset value of the firm and the expected return (drift rate) of the asset, and an increasing function of the volatility of the firms assets. However, the effect of time to maturity of the debt (31) is ambiguous. The probability of default will increase with time to maturity if

$$\frac{\ln(\frac{V}{F})}{2\sigma_V T^{3/2}} + \frac{\sigma_V}{4\sqrt{T}} > \frac{\mu_V}{2\sigma_V\sqrt{T}} \tag{32}$$

or

$$\mu_V < \frac{1}{T}\ln(\frac{V}{F}) + \frac{1}{2}\sigma_V \tag{33}$$

The table below shows the results from the comparative static analysis:

Table 2

A summary of the Comparative Statics for the objective probability of default

Comparative Statics	Effects on P
V	< 0
σ_V	> 0
μ_V	< 0
T	$\lesseqgtr 0$

.

4 Numerical examples - Implication for risk management

An example of the calculations presented in section 3 is elaborated in Appendix C. We here assume a firm with a market capitalization of \$3 bn ($E = 3$), an equity volatility of 80% per annum ($\sigma_E = 0.8$) and total liabilities of \$10 bn ($F = 10$). We further assume that all liabilities are due in one year ($T = 1$) and that the risk free interest rate is 5% ($r = 0.05$). In addition, the growth rate of equity is 10% ($\mu_E = 0.1$).

The asset value and volatility implied by the equity value, equity volatility and liabilities are calculated by solving the system (17) and (18) simultaneously. The risk adjusted and risk neutral default probability are calculated according to (4) and (7) respectively and finally the default probability sensitivities are found from (28),(29),(30) and (31).For details on the calculations, see Appendix C.

4.1 The sensitivity measures of the objective default probability

From the calculations in Appendix C, we now take a closer look at the default sensitivities w.r.t. the value of the firm, the volatility of the firm, the expected return on the firms asset and the time horizon of debt. We want to examine the behavior of these "Greeks" from our data and study when the respective sensitivity measures are at their maximum influence.

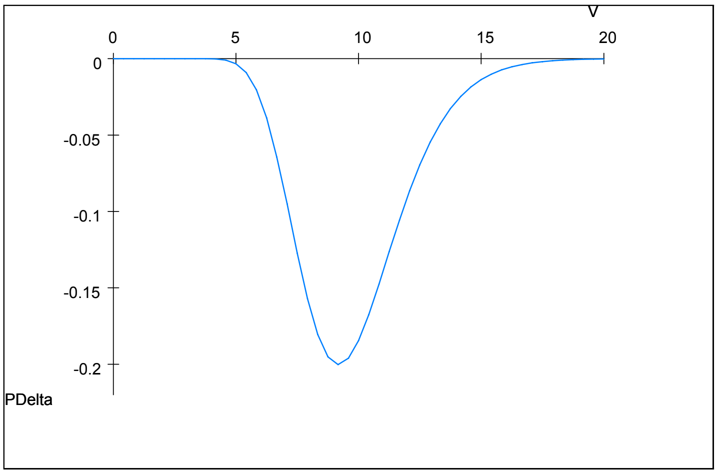

Figure 2: Sensitivity of the default probability to changes in the firms value.$F = 10, \sigma_V = 21.23\%, \mu_V = 6.32\%, T = 1$.

A closer investigation of the default probability delta of the example firm is given in Figure 2. Increased assets value always reduces the default probability: the delta is strictly $\leqslant 0$. This is what we should expect according to section 3. The highest sensitivity of the default probability for changes in firm value occur when the value of the firm is near the market value of debt, 9.40. When the market value of the firm is either far below or far above the value of debt, default is almost certain respespectively. almost impossible, so that marginal changes in firm value hardly have an effect on the default probability.

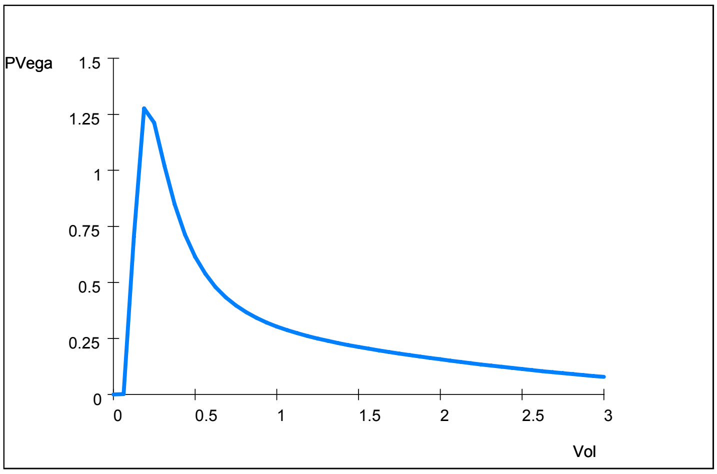

Figure 3: Sensitivity of the default probability to changes in the volatility of the firms value. $F = 10, V = 12.4, \mu_V = 6.32\%, T = 1$.

Figure 3 shows the sensitivity of the default probability to a change in the volatility of our company. An increased volatility of the assets always increases the default probability as expected. The default probability is most sensitive to changes in volatility when the volatility is around 17% for our case. For very high and very low volatilities, the probability vega is close to 0. For a given capital structure, a very low volatility will make the default probability very low and insensitive to marginal changes. For very high volatilities, the default probability will approach 100% already and is not further affected by marginal changes in the volatility of the firm.

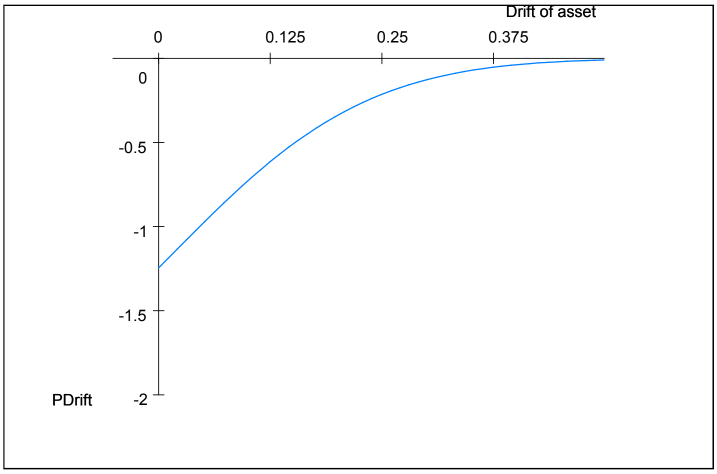

Figure 4: Sensitivity of the default probability to changes in the drift of the firms value. $F = 10, \sigma_V = 21.23\%, V = 12.4, T = 1$.

Figure 4 shows the sensitivity of the default probability for changes in the drift of the asset value (again for the example company). Increasing the drift of the assets, decreases the default probability. The default probability is most sensitive to changes in the drift of the asset for low drift levels. Note that drift levels below the risk free rate of return, although plotted, make no economic sense. For very high drift levels the effect is negligible as high drift makes default highly unlikely over a 1 year horizon, and hence the marginal effect is negligible.

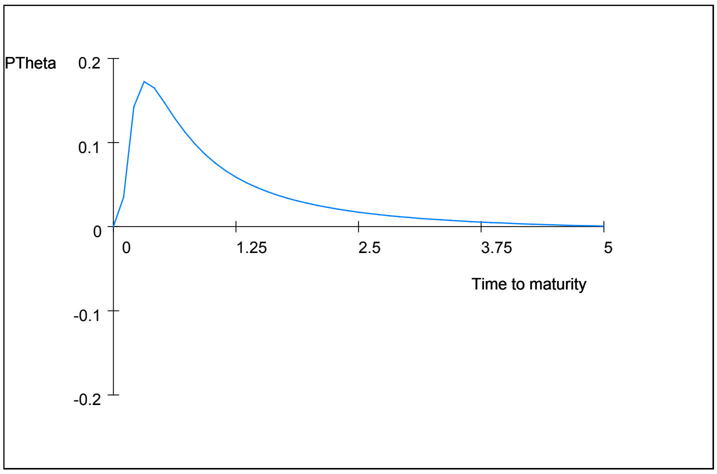

Figure 5: Sensitivity of the default probability to changes in the time horizone of debt $F = 10, \sigma_V = 21.23\%, \mu_V = 6.32\%, V = 12.4$

Figure 5 shows the sensitivity of the default probability for a change in the time horizon of debt. In our case, since $6.32\% < \frac{1}{1}\ln(\frac{12.4}{10}) + \frac{1}{2}0.2123 = 32.1\%$, increasing time horizon of debt will lead to higher default probability. This is usually the case, unless we face a company with very low equity ratio and high firm value volatility. For such firms the default probability is near 1 and the "only thing that can help" is an increasing time horizon together with high asset volatility, which make some scenarios possible where the asset value increases extremely and move the firm away from the default barrier.

Note also that the sensitivity measure w.r.t. time horizon is most sensitive for short time horizon (2-3 months in our case). For very small time horizons, the default probability tends to be closer and closer to zero (ref. earlier discussion) and is insensitive to marginal changes. A marginal increase in the time horizon for longer maturities will not affect the default probability much for our case.

4.2 The difference between the risk neutral and objective probability of default

As discussed earlier, it is important to analyze the difference between risk neutral and risk adjusted default probabilities in context of credit risk management. The panel below illustrates numerically the scenarios (in terms of parameter values) in which the differences between risk neutral and real probabilities are highlighted. Again, we have started out with the example from Appendix C. All panels have a risk free rate of 5% and a face value of debt of 10. In all panels, the values of equity are set to 1,3, and 10 respectively and the volatilities of equity are set to 40%, 80% and 150%. In panel 1 and 3 the time to maturity is set to 1 year, while in panel 2 and 4 it is set to 5 years. In panel 1 and 2, the drift of equity is set to 10% while in panel 3 and 4 it is set to 20%.The calculation of risk neutral and objective probabilities of default follow the set up from section 3.

Table 2
Representative values of risk neutral and risk adjusted probability of default

PANEL 1	$T = 1$	$\mu_E = 10\%$	$r = 5\%$	$F = 10$
E	σ_E	$N(-d_2)$	$N(-\hat{d}_2)$	$\frac{N(-d_2)-N(-\hat{d}_2)}{N(-d_2)}$
1	40%	0.47%	0.33%	29.8%
1	80%	15.53%	14.09%	9.3%
1	150%	61.75%	60.47%	2.1%
3	40%	0.25%	0.17%	32.0%
3	80%	12.70%	11.44%	9.9%
3	150%	56.05%	54.74%	2.3%
10	40%	0.03%	0.02%	33.3%
10	80%	7.14%	6.32%	11.5%
10	150%	44.68%	43.36%	3.0%

PANEL 2	$T = 5$	μ_E= 10%	$r = 5\%$	$F = 10$
E	σ_E	$N(-d_2)$	$N(-\hat{d}_2)$	$\frac{N(-d_2)-N(-\hat{d}_2)}{N(-d_2)}$
1	40%	21.14%	14.00%	33.8%
1	80%	74.72%	70.05%	6.3%
1	150%	97.82%	97.40%	0.4%
3	40%	17.21%	11.02%	35.97%
3	80%	68.41%	63.29%	7.5%
3	150%	96.19%	95.53%	0.7%
10	40%	9.80%	5.79%	40.9%
10	80%	56.67%	51.12%	9.8%
10	150%	93.08%	92.03%	1.2%
PANEL 3	$T = 1$	μ_E= 20%	$r = 5\%$	$F = 10$
E	σ_E	$N(-d_2)$	$N(-\hat{d}_2)$	$\frac{N(-d_2)-N(-\hat{d}_2)}{N(-d_2)}$
1	40%	0.47%	0.15%	68.1%
1	80%	15.53%	11.48%	26.1%
1	150%	61.75%	57.89%	6.3%
3	40%	0.25%	0.07%	72.0%
3	80%	12.70%	9.20%	27.6%
3	150%	56.05%	52.09%	7.1%
10	40%	0.03%	0.007%	76.7%
10	80%	7.14%	4.91%	31.2%
10	150%	44.68%	40.76%	8.8%
PANEL 4	$T = 5$	μ_E= 20%	$r = 5\%$	$F = 10$
E	σ_E	$N(-d_2)$	$N(-\hat{d}_2)$	$\frac{N(-d_2)-N(-\hat{d}_2)}{N(-d_2)}$
1	40%	21.14%	5.06%	76.1%
1	80%	74.72%	59.74%	20.1%
1	150%	97.82%	96.36%	1.5%
3	40%	17.21%	3.72%	78.4%
3	80%	68.41%	52.39%	23.4%
3	150%	96.19%	93.94%	2.3%
10	40%	9.80%	1.65%	83.2%
10	80%	56.67%	40.08%	29.3%
10	150%	93.08%	89.59%	3.7%

These panels shows some interesting phenomena. When equity increases, absolute default probability under both measures decreases. Keep in mind that we assume that the equity value (the call option value) can be found from the market and is thereof treated as a exogenous variable. In our model the probability of default is "back out", that is treated as endogenous variables solved in the model (the same is true for both for the risk neutral and the objective probability of default). When we here change the equity value partially, we have to resolve the hole model (all the calculation steps described in chapter 3). Changing the equity value, changes both the debt value and the asset value but the debt ratio (debt to total asset) will decrease. As the debt ratio decreases, the probability of hitting the default barrier decreases (that is when the market value of debt is below the face value of debt, in our case 10) What happens is that the distance to default has increased making default less likely. As our equity drift is higher than the risk free rate of return (10% versus 5% in panel 1,2 and 20% versus 5% in panel 3,4), the risk neutral default probability is always higher (ref. earlier discussion). Changing the equity value from 1 to 10 in our panel does not change the difference between the two measures significant. There is only a marginal increase in the difference between these measures as we increase the value of equity. For example, if we increase the value of equity from 3 to 10 at 40% equity volatility in panel 1, the relative difference between the risk neutral and objective probability of default increases from 32.0% to 33.3% (that is the risk neutral default probability is 32.0% resp. 33.3% higher than the objective default probability).

The difference between risk neutral and real default probabilities is significant for low values of the volatility of equity. As equity volatility increases, the difference drops significantly for all panels. The effect is increased by a longer time horizon. E.g. in panel 2 for an equity value of 10, the difference between the measures drops from 40.9% to 1.2% when we change the equity volatility from 40% to 150%. That is, when volatility of equity is at this level (150%), both risk neutral and objective default probability tend towards 1 and the difference is negligible. When the volatility of equity is lower (e.g. 40%) the default probability for both measures are lower. At these level, the drift effect has more to say. A small change in drift might increase the default probability dramatically at these level of equity volatility. The time horizon increases this effect. E.g., if we look at panel 4 (here the equity drift is 20% and the time horizon 5) at a equity value of 1, changing volatility from 40% to 150% reduces the difference between the probability measures from 76.1% to 1.5%.

When the time horizon increases, the difference between the measures can increase or decrease depending of which capital structure and equity volatility regime is prevailing. For low equity volatilities, increasing time horizon lowers the relative difference between the probabilities. For high equity volatilities, an increasing time horizon of debt leads to a lower difference for all panels. The effect is highest for low equity values. At low equity volatility the default probabilities are both close to 0, but as time increases, bankruptcy become more likely. This is in particular true for the risk neutral measure since the drift effect become more apparent. For very high equity volatilities, both measures will tend towards 1 as time horizon increases (unless the drift is very high) and the relative difference between the probability measures decreases

4.3 Implication for credit risk management

Delianides and Geske (2003) shows that risk neutral default probabilities could have some predictive power for risk adjusted default frequencies and thereof be used in credit risk management although they stress the fact that the main use of risk neutral probabilities are for pricing purposes (KMV, as mentioned, use directly these ideas). In particular they demonstrate that the risk neutral default probabilities estimated from both Merton (1974) and Geske (1977) successfully categorize/predict firms credit rating which in turn affect their market values[12] These probabilities (although not representative for the "real" probability of

[12] In particular they estimate risk neutral default probabilities from these models using monthly time series from 1988 to 1999 for the firms equity, rolling volatility, debt maturity and so forth. They then examine the changes in these default probabilties before the event of a rating migration or default.

default) possess significant early information about credit rating migration. The use of risk neutral default probabilities is used because (as this paper clearly shows) these measures are much easier to estimate than risk adjusted default probabilities because they do not require estimation of the firm's (or any asset's) expected return.

Risk neutral and "real" default probabilities have the same sign of the sensitivities w.r.t. the underlying parameters. This is also discussed in Delianides and Geske (2003). That is, if you what to i.e. study the impact on increasing volatility of the firms equity, both the risk neutral and objective probability will be affected the same way by an increase in the probability of default Default probability sensitivities is a valuable tool for a risk manager as one is able to perform what-if analysis on the credit risk with respect to changes in the capital structure and business risk of a firm.

We state that risk neutral default probability should be treated when caution. . Risk neutral probabilities though tend to be very high for "high credit quality" firms (low equity volatility and high equity to total asset ratio). As we already have pinpointed, this might lead to serious mistakes when allocating scare capital. An overestimation of real default probabilities for investment grade bonds and loans with high credit quality might lead the bank to allocate less capital to such projects and destroy value created for the stockholders of the bank.

5 Conclusions and further work

In conclusion, option based default probability estimation represent an innovative approach for measuring and managing credit risk. Banks can no longer ignore equity market information in their Credit VAR and RAROC applications. They need to monitor equity market valuations constantly along with its volatility and changes in capitial structure of the firm and interpret implication for credit risk. The popularity of the KMV approach in the industry which is based on the BSM model, understates this importance.

In any Credit risk management model, good estimates of default probabilities are essential. Although risk neutral default probabilities from BSM gives valuable information, they can give quite misleading information when calculating risk adjusted performance such as RAROC for instance. This paper has described a procedure to derive real or objective (as opposed to risk neutral) default probabilities in a BSM framework. Typically we demonstrate that for investment graded bonds (companies with low equity volatility and low debt ratio) the risk neutral default probabilities significantly overestimate the true probability of default. This leads to a lower RAROC figure for such firms and we might reject counterparts to the bank that in reality would have contributed to value creation with RAROC above the hurdle rate (the return on equity requirement).

The probability of default increases with firms volatility and decreases with the firms drift and value. The effect of time to maturity has a ambigious effect on the default probability depending on the relationship between the volatility, the drift and the face value of debt. Although the numerical experiments are too limited to allow general conclusions, they indicate that the capital structure and risk/returns characteristics of the firm may have a strong combined effect on default probability. Sensitivity analysis is an important element in monitoring credit risk. This provides the credit analyst with the possibility perform what-if analyses regarding the model's various inputs such as changes in the equity valuation, capital structure and business risk of the firm. We suggest to use these sensitivity measures for the default probability as an objective examination and use them in combination with other sources of information in making credit risk decisions.

We point out two direction for further work. First this type of framework we have shown in the paper could also be done for more advanced structural models created after BSM as mention in the article. Second, the comparative statics of this default probability, or default Greeks, give also a theoretical basis for empirical analyses of default probability. Since the majority of empirical default studies is neither guided nor restricted by financial theory, the results in this paper may be a valuable contribution to such studies.

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Appendix A: Sensitivity measures for the equity

- Equity Gamma measures the change in the equity value w.r.t. a change in the equity delta and is given by::

$$\begin{aligned}
\Gamma^E &= \frac{\partial^2 E}{\partial V^2} \\
&= \frac{\partial[N(d_1)]}{\partial V} = n(d_1) \cdot \frac{\partial(d_1)}{\partial V} \\
&= n(d_1) \cdot \frac{1}{(\frac{V}{F})} \cdot \frac{1}{F} \cdot \frac{1}{\sigma\sqrt{T}} \\
&= \frac{n(d_1)}{V\sigma\sqrt{T}} > 0
\end{aligned}$$

- Equity Theta $\Theta^E = \frac{\partial E}{\partial t}$. Theta measures the time decay or the rate of change of the equity value w.r.t. the passage of time, all other things equal. This means that ∂t is always negative, so that we can write $\Theta^E = -\frac{\partial E}{\partial t}$.First we give some elements that will be used in the calculations of equity theta:

$$\begin{aligned}
d_2 &= d_1 - \sigma\sqrt{T} \\
d_2^2 &= d_1^2 - 2d_1\sigma\sqrt{T} + \sigma^2 T \\
&= d_1^2 - 2[\ln(\frac{V}{F}) + (r + \frac{1}{2}\sigma^2)T] + \sigma^2 T \\
&= d_1^2 - 2\ln(\frac{Ve^{rT}}{F})
\end{aligned}$$

Next:

$$n(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} + \ln(\frac{Ve^{rT}}{F})}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} e^{\ln(\frac{Ve^{rT}}{F})} = n(d_1)\frac{Ve^{rT}}{F}$$

Similarly:

$$n(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} = n(d_2)\frac{F}{Ve^{rT}}$$

Finally:

$$\frac{\partial d_2}{\partial T} - \frac{\partial d_1}{\partial T} = [-\frac{\ln(\frac{V}{F})}{2\sigma T^{3/2}} + \frac{r}{2\sigma\sqrt{T}} - \frac{\sigma}{4\sqrt{T}}] - [-\frac{\ln(\frac{V}{F})}{2\sigma T^{3/2}} + \frac{r}{2\sigma\sqrt{T}} + \frac{\sigma}{4\sqrt{T}}] = -\frac{\sigma}{2\sqrt{T}}$$

The equity theta can now be found as:

$$\Theta^E = -\frac{\partial E}{\partial t}$$

$$= -\left\{ V\frac{\partial N(d_1)}{\partial T} + \underbrace{\frac{\partial V}{\partial T}}_{=0} N(d_1) + rFe^{-rT}N(d_2) - Fe^{-rT}\frac{\partial N(d_2)}{\partial T} \right\}$$

$$= -\left\{ Vn(d_1)\frac{\partial d_1}{\partial T} + rFe^{-rT}N(d_2) - Fe^{-rT}n(d_2)\frac{\partial d_2}{\partial T} \right\}$$

Substituting and rearranging terms we get:

$$= -\left\{ Vn(d_1)\frac{\partial d_1}{\partial T} - Fe^{-rT}n(d_1)\frac{Ve^{rT}}{F}\frac{\partial d_2}{\partial T} + rFe^{-rT}N(d_2) \right\}$$

$$= -\left\{ Vn(d_1)[\frac{\partial d_1}{\partial T} - \frac{\partial d_2}{\partial T}] + rFe^{-rT}N(d_2) \right\}$$

$$= -\left\{ Vn(d_1)[\frac{\sigma}{2\sqrt{T}}] + rFe^{-rT}N(d_2) \right\}$$

$$= -\frac{Vn(d_1)\sigma}{2\sqrt{T}} - rFe^{-rT}N(d_2)$$

Appendix B: Derivation of Probability Greeks

- The sensitivity of the default probability with respect to a change in the value of the firm V is:

$$\frac{\partial p}{\partial V} = \frac{\partial N(-\hat{d}_2)}{\partial V} = n(-\hat{d}_2)\cdot\frac{\partial(\hat{d}_2)}{\partial V}$$

$$= n(-\hat{d}_2)\cdot\frac{\partial\left[-\left(\frac{\ln(\frac{V}{F})+(\mu_V-\frac{\sigma_V^2}{2})T}{\sigma_V\sqrt{T}}\right)\right]}{\partial V}$$

$$= n(-\hat{d}_2)\cdot\frac{\partial}{\partial V}[\frac{-\ln(\frac{V}{F})}{\sigma_V\sqrt{T}}] = -n(\hat{d}_2)\cdot\frac{1}{V\sigma_V\sqrt{T}}] < 0$$

- The sensitivity of the default probability with respect to a change in the volatility of the value of the firm σ_V:

$$\frac{\partial p}{\partial \sigma_V} = \frac{\partial N(-\hat{d}_2)}{\partial \sigma_V} = n(-\hat{d}_2)\cdot\frac{\partial(-\hat{d}_2)}{\partial \sigma_V}$$

$$= n(\hat{d}_2)\cdot\frac{\partial\left[-\left(\frac{\ln(\frac{V}{F})+(\mu_V-\frac{\sigma_V^2}{2})T}{\sigma_V\sqrt{T}}\right)\right]}{\partial \sigma_V}$$

$$= -n(\hat{d}_2)\cdot\frac{\partial\left[\frac{\ln(\frac{V}{F})}{\sigma_V\sqrt{T}} + \frac{\mu_V\sqrt{T}}{\sigma_V} - \frac{\sigma_V}{2}\sqrt{T}\right]}{\partial \sigma_V}$$

$$= -n(\hat{d}_2)\cdot\left[-\frac{\ln(\frac{V}{F})}{\sigma_V^2\sqrt{T}} - \frac{\mu_V\sqrt{T}}{\sigma_V^2} - \frac{1}{2}\sqrt{T}\right]$$

$$= n(\hat{d}_2)\cdot\left[\frac{\ln(\frac{V}{F}) + T\mu_V + \frac{1}{2}T\sigma_V^2}{\sigma_V^2\sqrt{T}}\right] > 0$$

- The sensitivity of the default probability with respect to a change in the drift of the firm μ_V:

$$\frac{\partial p}{\partial \mu_V} = \frac{\partial N(-\hat{d}_2)}{\partial \mu_V} = n(-\hat{d}_2)\cdot\frac{\partial(\hat{d}_2)}{\partial \mu_V}$$

$$= n(\stackrel{\wedge}{d}_2) \cdot \frac{\partial \left[- \left(\frac{\ln(\frac{V}{F}) + (\mu_V - \frac{\sigma_V^2}{2})T}{\sigma_V \sqrt{T}} \right) \right]}{\partial \mu_V}$$

$$= -n(\stackrel{\wedge}{d}_2) \cdot \frac{\sqrt{T}}{\sigma_V} < 0$$

- The sensitivity of the default probability with respect to a change in the time to maturity T:

$$\frac{\partial p}{\partial T} = \frac{\partial N(-\stackrel{\wedge}{d}_2)}{\partial T} = n(-\stackrel{\wedge}{d}_2) \cdot \frac{\partial (\stackrel{\wedge}{d}_2)}{\partial T}$$

$$= n(\stackrel{\wedge}{d}_2) \cdot \frac{\partial \left[- \left(\frac{\ln(\frac{V}{F}) + (\mu_V - \frac{\sigma_V^2}{2})T}{\sigma_V \sqrt{T}} \right) \right]}{\partial T}$$

$$= -n(\stackrel{\wedge}{d}_2) \cdot \frac{\partial \left[\frac{\ln(\frac{V}{F})}{\sigma_V \sqrt{T}} + \frac{\mu_V \sqrt{T}}{\sigma_V} - \frac{\sigma_V}{2} \sqrt{T} \right]}{\partial T}$$

$$= -n(\stackrel{\wedge}{d}_2) \cdot \left[- \frac{\ln(\frac{V}{F})}{2 \sigma_V T^{3/2}} + \frac{\mu_V}{2 \sigma_V \sqrt{T}} - \frac{\sigma_V}{4 \sqrt{T}} \right]$$

$$= n(\stackrel{\wedge}{d}_2) \cdot \left[\frac{\ln(\frac{V}{F})}{2 \sigma_V T^{3/2}} - \frac{\mu_V}{2 \sigma_V \sqrt{T}} + \frac{\sigma_V}{4 \sqrt{T}} \right] \gtrless 0$$

Appendix C: Numerical example of the framework

Consider the example of a firm with a market capitalization of \$3 bn ($E = 3$), an equity volatility of 80% per annum ($\sigma_E = 0.8$) and total liabilities of \$10 bn ($F = 10$). Assume further that all liabilities are due in one year ($T = 1$) and that the risk free interest rate is 5% ($r = 0.05$). The asset value and volatility implied by the equity value, equity volatility and liabilities are calculated by solving the system (17) and (18) simultaneously.

$$V \cdot N\left(\frac{\ln(V/10)+(0.05+\sigma_V^2/2)\cdot 1}{\sigma_V\cdot\sqrt{1}}\right) - 10\cdot e^{-0.05\cdot 1}\cdot N\left(\frac{\ln(V/10)+(0.05-\sigma_V^2/2)\cdot 1}{\sigma_V\cdot\sqrt{1}}\right) - 3 = 0$$

$$0.8\cdot 3 - \sigma_V\cdot V\cdot N\left(\frac{\ln(V/10)+(0.05+\sigma_V^2/2)\cdot 1}{\sigma_V\cdot\sqrt{1}}\right) = 0$$

This gives an implied market value of the firms assets of V =\$12.4 bn and an implied asset volatility of σ_V =21.3%. This implies a market value of debt of $D = V - E$ = \$12.4bn−\$3bn= \$9.4bn. Having found the value of the firm and its volatility, the risk-neutral probability of default can be calculated as[13]:

$$N(-d_2) = N\left(-\left\{\frac{\ln\frac{12.4}{10} + (0.05 - \frac{0.213^2}{2})1}{0.213\sqrt{1}}\right\}\right) = 12.8\%$$

In addition, assume that the β of the company is estimated to be 1.3 and the expected return on the market portfolio μ_M is 9% (4% risk premium). According to the CAPM $\mu_E = 0.05 + 1.3(0.09 - 0.05) = 0.1$. Thus, the the drift of equity is found to be 10%. The equity Delta, Gamma and Theta can be calculated as:

$$\begin{aligned}
\Delta^E &= N(d_1) = 0.91228 \\
\Gamma^E &= \frac{n(d_1)}{V\sigma\sqrt{T}} = 0.06052 \\
\Theta^E &= -\frac{Vn(d_1)\sigma}{2\sqrt{T}} - rFe^{-rT}N(d_2) = -0.62511
\end{aligned}$$

Using equation (24) we can calculate the drift of the assets:

$$\begin{aligned}
\mu_V &= \frac{\mu_E E - \Theta^E - \frac{1}{2}\sigma_V^2 V^2\Gamma^E}{V\Delta^E} \\
&= \frac{0.10\cdot 3 + 0.62511 - (\frac{1}{2}\cdot 0.2123^2\cdot 12.4^2\cdot 0.06052)}{12.4\cdot 0.91228} = 0.063241 = 6.32\%
\end{aligned}$$

Having found V, σ_V and μ_V, we can now calculate the objective default probability:

$$\begin{aligned}
N(-d_2) &= N\left(-\frac{\ln\frac{V}{F} + (\mu_V - \sigma_V^2/2)T}{\sigma_V\sqrt{T}}\right) \\
&= N\left(-\frac{\ln\frac{12.4}{10} + (0.063241 - 0.2123^2/2)1}{0.2123\sqrt{1}}\right) \\
&= 0.11410 = 11.4\%
\end{aligned}$$

[13] In addtion, we could calculate the expected recovery rate, default spread and other related measures.

Note that the real (or objective) probability of default is lower than the risk neutral probability of default (11.4% compared to 12.8%).

The sensitivities of the default probability (default Greeks) are given by:

$$\frac{\partial p}{\partial V} = -n(\frac{\ln \frac{12.4}{10} + (0.063241 - 0.2123^2/2)1}{0.2123\sqrt{1}}) \cdot \frac{1}{12.4 \cdot 0.2123\sqrt{1}} = -0.073$$

$$\frac{\partial p}{\partial \sigma_V} = n(\frac{\ln \frac{12.4}{10} + (0.063241 - 0.2123^2/2) \cdot 1}{0.2123\sqrt{1}}) \cdot ..$$
$$..(\frac{\ln(\frac{12.4}{10}) + 1 \cdot 0.063241 + \frac{1}{2} \cdot 1 \cdot 0.2123^2}{0.2123^2\sqrt{1}}) = 1.289$$

$$\frac{\partial p}{\partial \mu_V} = -n(\frac{\ln \frac{12.4}{10} + (0.063241 - 0.2123^2/2)1}{0.2123\sqrt{1}}) \cdot \frac{\sqrt{1}}{0.2123} = -0.909$$

$$\frac{\partial p}{\partial T} = n(\frac{\ln \frac{12.4}{10} + (0.063241 - 0.2123^2/2)1}{0.2123\sqrt{1}}) \cdot ..$$
$$..(\frac{\ln(\frac{12.4}{10})}{2 \cdot 0.2123 \cdot 1^{3/2}} - \frac{0.063241}{2 \cdot 0.2123\sqrt{1}} + \frac{0.2123}{4\sqrt{1}}) = 0.0793$$

In this example, an increase in the value of the firm by, say \$0.1, decreases the default probability with 0.1·0.073=0.73%. An increase in the volatility of the firm by 1%, increases the default probability with 0.01·1.289=1.289%. Increasing the drift by 1%, decreases the default probability by 0.01·0.909=0.909%. Decreasing the time horizon of debt with 1 month (1/12), decreases the default probability with (1/12)·0.0793=0.66%. Note that we are looking at partial derivatives so that the changes are only valid for small changes in the underlying parameters.