Scenario-Free Approximations to Stochastic Programming via Decision Rules

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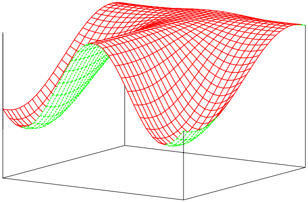
Deterministic Optimisation

Deterministic optimisation model:

$$\begin{array}{ll} \underset{x}{\text{minimise}} & f(x, \xi) \\ \text{subject to} & x \in \mathcal{X}(\xi), \end{array}$$



where

- x are *decision variables*
- ξ are (precisely known) *parameters*

Real world is *uncertain*. Why not use $\xi = \mathbb{E}[\,\tilde{\xi}\,]$?

Deterministic Optimisation

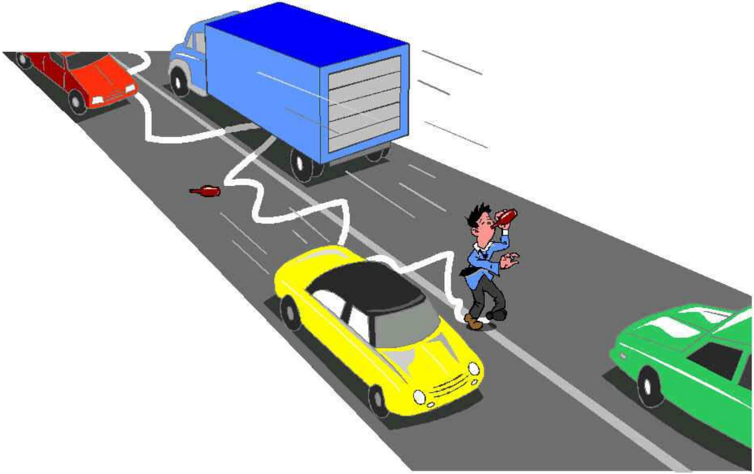
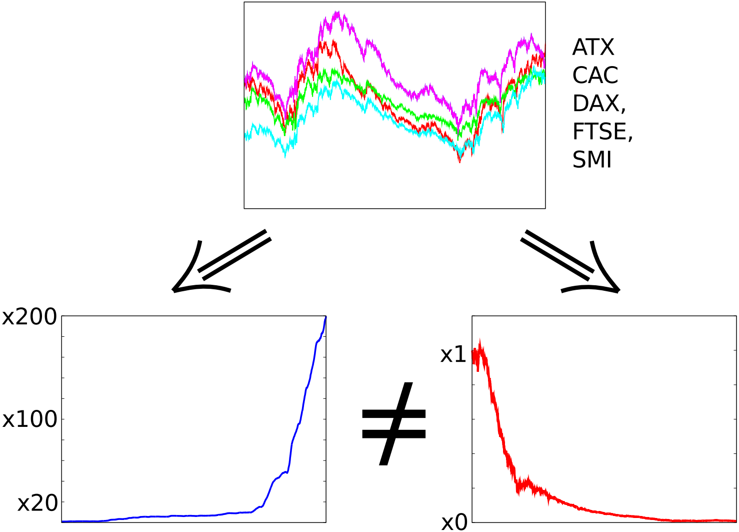


Image Source: Sam L. Savage, Stanford University

$\text{alive}\left(\mathbb{E}\left[\text{position}\right]\right) = \text{true}$, but $\mathbb{E}\left[\text{alive}\left(\text{position}\right)\right] = \text{false}$!

Deterministic Optimisation

Portfolio optimisation:

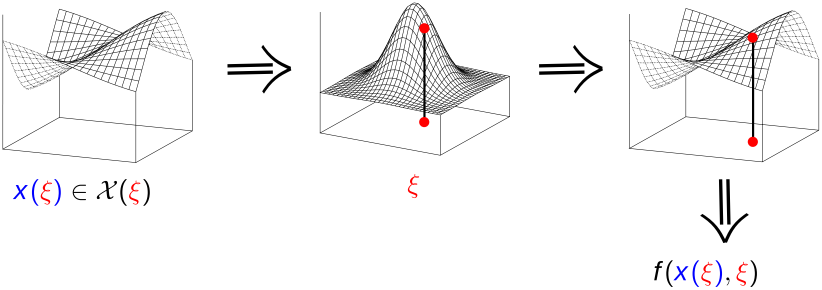


$\text{wealth}\left(\mathbb{E}\left[\text{stock returns}\right]\right)$ vs. $\mathbb{E}\left[\text{wealth}\left(\text{stock returns}\right)\right]$

Stochastic Programming

One-stage stochastic program:

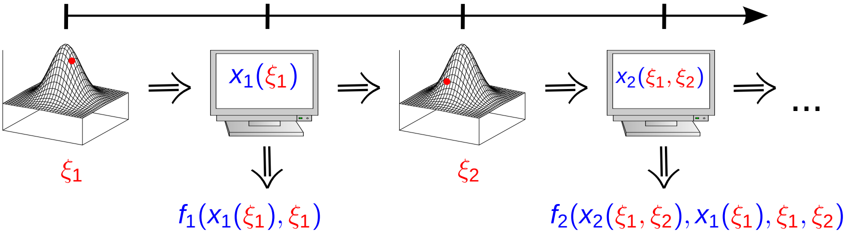
$$\begin{array}{ll} \underset{x(\cdot)}{\text{minimise}} & \mathbb{E}\left[f(x(\xi),\xi)\right] \\ \text{subject to} & x(\xi) \in \mathcal{X}(\xi) \quad \mathbb{P}\text{-a.s.} \end{array}$$



Stochastic Programming

Multi-stage stochastic program: *several* sequential decisions

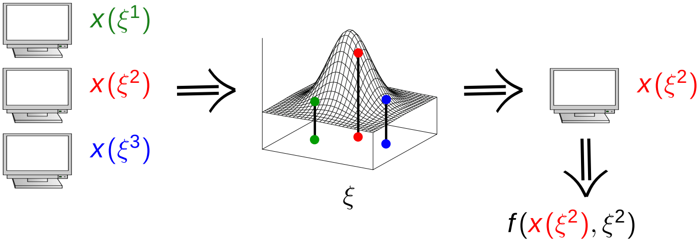
- capacity expansion: several investment stages
- production planning: annual production plan (seasonalities)
- portfolio optimisation: rebalancing, asset & liability mgmt.
- ...



Scenario-Based S.P.

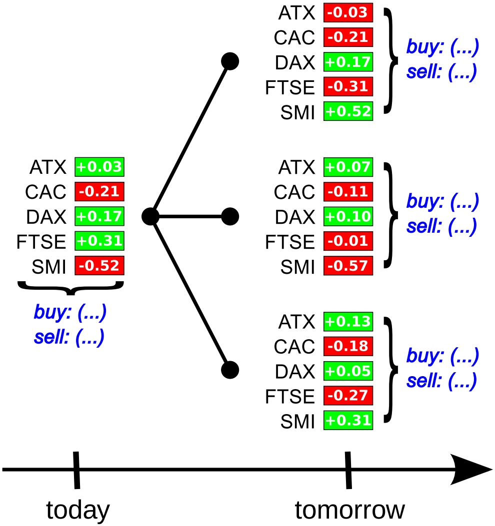
Discretise distribution: into *scenarios*

$$\begin{aligned} \underset{x^s}{\text{minimise}} \quad & \sum_{s \in \mathcal{S}} p_s f(x^s, \xi^s) \\ \text{subject to} \quad & x^s \in \mathcal{X}(\xi^s) \quad \forall s \in \mathcal{S}. \end{aligned}$$



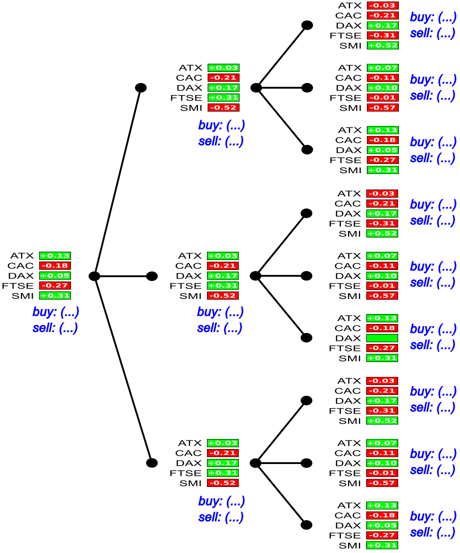
Scenario-Based S.P.

Portfolio optimisation:

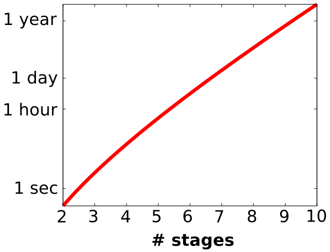


Scenario-Based S.P.

Multi-stage stochastic program:

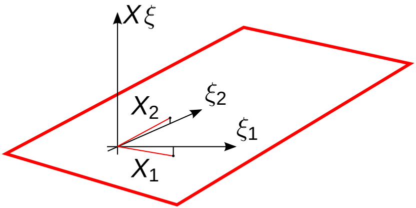


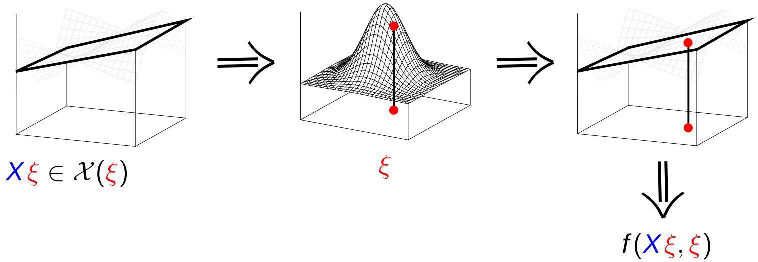
Solution time:



Linear Decision Rules[1]

$$\begin{array}{ll} \underset{X}{\text{minimize}} & \mathbb{E}\left[f(X\xi, \xi)\right] \\ \text{subject to} & X\xi \in \mathcal{X}(\xi) \quad \mathbb{P}\text{-a.s.} \end{array}$$





[1]Ben-Tal *et al.*, Math. Programming, 2004.

Linear Decision Rules

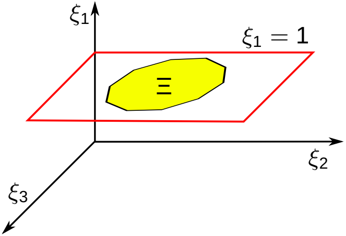
Linear S.P. with fixed recourse:

$$\begin{array}{ll} \underset{x(\cdot),s(\cdot)}{\text{minimize}} & \mathbb{E}\left(c(\xi)^\top x(\xi)\right) \\ \text{subject to} & Ax(\xi) + s(\xi) = b(\xi) \quad \mathbb{P}\text{-a.s.} \\ & s(\xi) \geq 0 \quad \mathbb{P}\text{-a.s.} \end{array}$$

$\Longrightarrow$ Optimize over all measurable functions $x(\cdot)$ and $s(\cdot)$

Assumptions:

- Linear data dependence: $c(\xi) = C\xi$, $b(\xi) = B\xi$
- Tractable uncertainty set: $\operatorname{supp}\mathbb{P} = \Xi$, $\quad \xi \in \Xi \Rightarrow \xi_1 = 1$
- Finite second moments: $M = \mathbb{E}(\xi\xi^\top)$



Linear Decision Rules

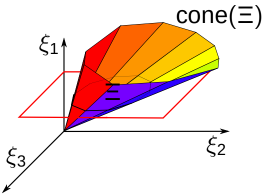
Linearize the decisions:

$$\begin{aligned} \underset{X,S}{\text{minimize}} \quad & \mathbb{E}\left(\xi^\top C^\top X \xi\right) \\ \text{subject to} \quad & AX\xi + S\xi = B\xi \quad \mathbb{P}\text{-a.s.} \\ & S\xi \geq 0 \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

$\Longrightarrow$ Optimize over all matrices X and S

Reexpress the semi-infinite constraints:

$$\begin{aligned} \underset{X,S}{\text{minimize}} \quad & \operatorname{tr}\left(MC^\top X\right) \\ \text{subject to} \quad & AX + S = B \\ & S \in (\text{cone}(\Xi)^*)^m \end{aligned}$$



$\Longrightarrow$ Conic program of polynomial size in the input data

Semi-Infinite Constraints

Lemma

For any $z \in \mathbb{R}^k$ the following statements are equivalent:

(i) $z^\top \xi \geq 0 \ \forall \, \xi \in \Xi$

(ii) $z \in cone(\Xi)^*$

(iii) $\exists \, \lambda \in \mathbb{R}^l$ *with* $\lambda \geq 0$, $W^\top \lambda = z$, *and* $h^\top \lambda \geq 0$

Proof.

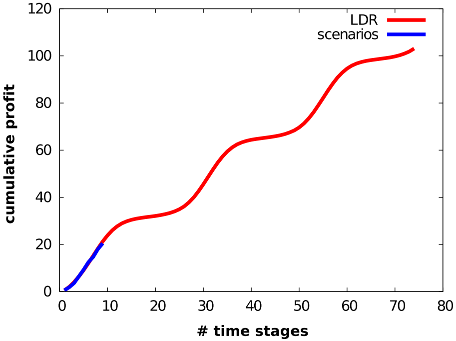
$$
\begin{aligned}
& z^\top \xi \geq 0 \text{ for all } \xi \text{ subject to } W\xi \geq h \\
\iff \quad & 0 \leq \min_{\xi \in \mathbb{R}^k} \left\{ z^\top \xi : W\xi \geq h \right\} \\
\iff \quad & 0 \leq \max_{\lambda \in \mathbb{R}^l} \left\{ h^\top \lambda : W^\top \lambda = z, \, \lambda \geq 0 \right\} \\
\iff \quad & \exists \, \lambda \in \mathbb{R}^l \text{ with } W^\top \lambda = z, \, h^\top \lambda \geq 0, \, \lambda \geq 0
\end{aligned}
$$

□

Example

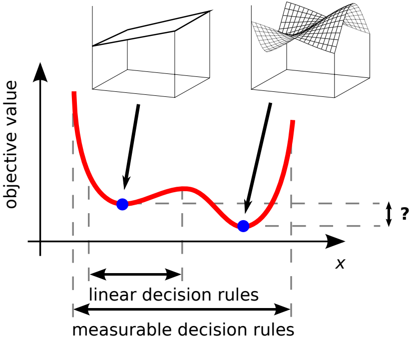
Inventory problem:

- three factories produce single good, one warehouse
- limited per-period production and storage capacities
- demand uniformly distributed around known nominal demand
- nominal demand seasonal: $\mathbb{E}(\xi_t) = 1{,}000 \times \left(1 + \frac{1}{2} \sin\left[\frac{\pi(t-1)}{12}\right]\right)$



Bounding the Optimality Gap[2]

So far: obtained upper bound on optimal value via restriction from measurable to linear decision rules

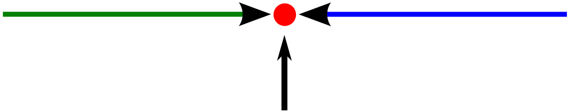


Idea: obtain lower bound on optimal value to bound suboptimality

[2]Kuhn *et al.*, Math. Programming, 2010.

Bounding the Optimality Gap

Step 1: dualize stochastic program in measurable decision rules



duality theory: under mild regularity conditions,
optimal values of primal and dual problems coincide!

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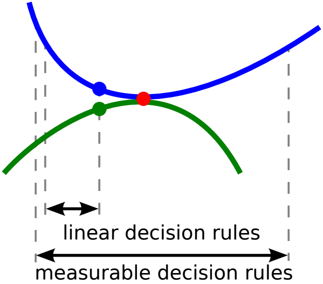
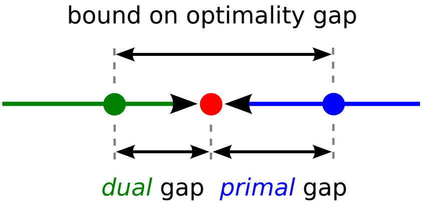
Result: dual stochastic program in measurable decision rules

Bounding the Optimality Gap

Step 2: restrict dual stochastic program to linear decision rules





Result: dual stochastic program with linear decision rules allows us to bound incurred suboptimality

Dual Linear Decision Rules

Min-max reformulation of linear S.P.:

$$\begin{array}{ll} \underset{x(\cdot),s(\cdot)}{\text{minimize}} & \underset{y(\cdot)}{\sup}\ \mathbb{E}\left(c(\xi)^\top x(\xi) + y(\xi)^\top [Ax(\xi) + s(\xi) - b(\xi)]\right) \\ \text{subject to} & s(\xi) \geq 0 \quad \mathbb{P}\text{-a.s.} \end{array}$$

$\Longrightarrow$ Minimize (supremum over all measurable dual decisions $y(\cdot)$) over all measurable primal decisions $x(\cdot)$ and $s(\cdot)$

Linearize the dual decisions:

$$\begin{array}{ll} \underset{x(\cdot),s(\cdot)}{\text{minimize}} & \underset{Y}{\sup}\ \mathbb{E}\left(c(\xi)^\top x(\xi) + \xi^\top Y^\top [Ax(\xi) + s(\xi) - b(\xi)]\right) \\ \text{subject to} & s(\xi) \geq 0 \quad \mathbb{P}\text{-a.s.} \end{array}$$

Dual Linear Decision Rules

Dual LDR problem is equivalent to:

$$\begin{aligned} \underset{x(\cdot),s(\cdot)}{\text{minimize}} \quad & \mathbb{E}\left(c(\xi)^\top x(\xi)\right) \\ \text{subject to} \quad & \mathbb{E}\left([Ax(\xi) + s(\xi) - b(\xi)]\,\xi^\top\right) = 0 \\ & s(\xi) \geq 0 \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

$\Longrightarrow$ constraint aggregation

And equivalent to:

$$\begin{aligned} \underset{X,S}{\text{minimize}} \quad & \text{tr}\left(MC^\top X\right) \\ \text{subject to} \quad & AX + S = B \\ & SM \in \text{cone}(\Xi)^m \end{aligned}$$

$\Longrightarrow$ Conic program of polynomial size in the input data

Semi-Infinite Constraints

Lemma

For any $z \in \mathbb{R}^k$ the following statements are "essentially" equivalent:

(i) $z \in cone(\Xi)$;

(ii) $\exists\, s(\cdot)$ *with* $\mathbb{E}(s(\xi)\,\xi) = z$ *and* $s(\xi) \geq 0$ $\mathbb{P}$*-a.s.*

Sketch of Proof.

The feasible sets of the conditions (i) and (ii) represent pointed cones in $\mathbb{R}^k$ $\Rightarrow$ w.l.o.g. assume that $z_1 = 1$. Thus,

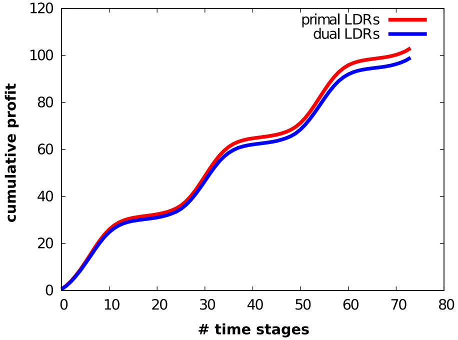
- (i) $\iff$ $z \in \Xi$
- (ii) $\iff$ z is the mean of the distribution $\mathbb{Q}$ with density $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}(\xi) = s(\xi)$
- As $\mathbb{Q}$ is supported on Ξ, (ii) is "essentially" equivalent to (i)

□

Example Problem Revisited

Inventory problem:

- three factories produce single good, one warehouse
- limited per-period production and storage capacities
- demand uniformly distributed around known nominal demand
- nominal demand seasonal: $\mathbb{E}(\xi_t) = 1{,}000 \times \left(1 + \frac{1}{2}\sin\left[\frac{\pi(t-1)}{12}\right]\right)$

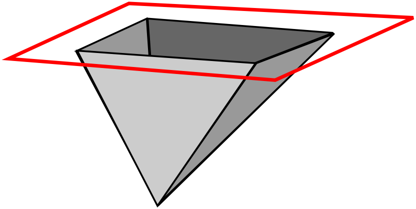


Piecewise Linear Decision Rules[3]

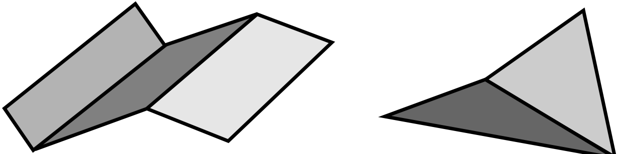
Linear decision rules can fail:

$$\begin{array}{lll} \underset{x(\cdot)}{\text{minimize}} & \mathbb{E}\left[x(\xi)\right] & \\ \text{subject to} & x(\xi) \geq \|\xi\|_1 & \mathbb{P}\text{-a.s.} \end{array}$$

where $\xi \sim \mathcal{U}[-1,1]^k$.



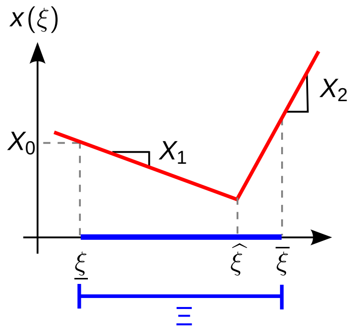
Remedy: use *piecewise linear* decision rules instead

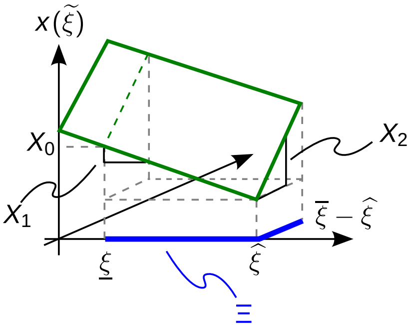


[3]Goh & Sim, Oper. Research, 2010; Georghiou *et al.*, Optimization Online, 2010.

Piecewise Linear Decision Rules

Piecewise linear decision rule $\equiv$ linear decision rule in *lifted* space:





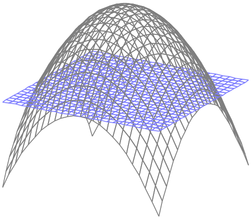
$$x(\xi) = X_0 + X_1\left(\min\left\{\xi, \widehat{\xi}\right\} - \underline{\xi}\right) + X_2\left(\max\left\{\xi, \widehat{\xi}\right\} - \widehat{\xi}\right)$$

$$x(\widetilde{\xi}) = X_0 + X_1\widetilde{\xi}_1 + X_2\widetilde{\xi}_2$$

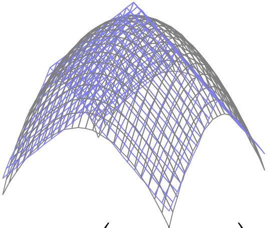
$$\text{where } \widetilde{\xi} = \begin{pmatrix} \min\left\{\xi, \widehat{\xi}\right\} - \underline{\xi} \\ \max\left\{\xi, \widehat{\xi}\right\} - \widehat{\xi} \end{pmatrix}$$

Increase Flexibility of Decision Rules

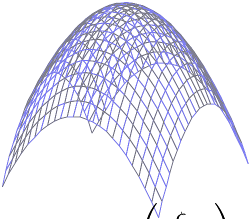
$$x(\xi) = \sum_{i=1}^{k'} x^i L_i(\xi), \quad \text{where} \quad L(\xi) = (L_1(\xi), \ldots, L_{k'}(\xi))^\top$$



$$L(\xi) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$



$$L(\xi) = \begin{pmatrix} \xi_1 \\ \max\{\xi_2, 0\} \\ \min\{\xi_2, 0\} \\ \max\{\xi_3, 0\} \\ \min\{\xi_3, 0\} \end{pmatrix}$$



$$L(\xi) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ (\xi_2)^2 \\ (\xi_3)^2 \end{pmatrix}$$

Lifting and Retraction Operators

Lifting: $L : \mathbb{R}^k \to \mathbb{R}^{k'}, \quad \xi \mapsto \xi'$

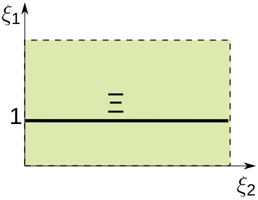
Retraction: $R : \mathbb{R}^{k'} \to \mathbb{R}^k, \quad \xi' \mapsto \xi$

(A1) L is continuous and satisfies $L_1(\xi) = 1$ for all $\xi \in \Xi$;

(A2) R is linear;

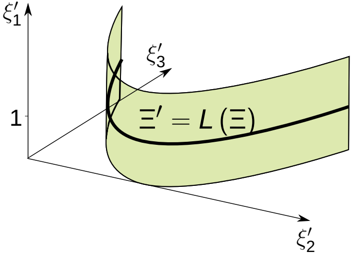
(A3) $R \circ L = \mathbb{I}_k$;

(A4) The component mappings of L are linearly independent



$$L \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ (\xi_2)^2 \end{pmatrix}$$

$$R \begin{pmatrix} \xi'_1 \\ \xi'_2 \\ \xi'_3 \end{pmatrix} = \begin{pmatrix} \xi'_1 \\ \xi'_2 \end{pmatrix}$$



Lifted Stochastic Program

- Probability distribution on lifted space

$$\mathbb{P}_{\xi'}(B') := \mathbb{P}_{\xi}\left(\{\xi \in \mathbb{R}^k : L(\xi) \in B'\}\right) \qquad \forall B' \in \mathcal{B}(\mathbb{R}^{k'}).$$

$$\begin{array}{lll} \min & \mathbb{E}_{\xi}\left(c(\xi)^\top x(\xi)\right) \\ \text{s.t.} & x \in \mathcal{L}_{k,n} \\ & Ax(\xi) \le b(\xi) \ \mathbb{P}_{\xi}\text{-a.s.} \\ & \hfill (\mathcal{SP}) \end{array} \quad \xrightarrow{\text{Lifting}} \quad \begin{array}{lll} \min & \mathbb{E}_{\xi'}\left(c(R\xi')^\top x(\xi')\right) \\ \text{s.t.} & x \in \mathcal{L}_{k',n} \\ & Ax(\xi') \le b(R\xi') \ \mathbb{P}_{\xi'}\text{-a.s.} \\ & \hfill (\mathcal{LSP}) \end{array}$$

Theorem

Problems $\mathcal{SP}$ and $\mathcal{LSP}$ are equivalent.

Decision Rule Approximations

$$x(\xi) = X'L(\xi)$$

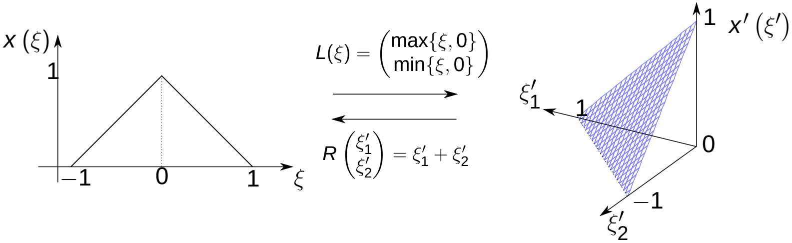
$$\begin{array}{ll} \min & \mathbb{E}_{\xi}\left(c(\xi)^{\top} X'L(\xi)\right) \\ \text{s.t.} & X' \in \mathbb{R}^{n \times k'} \\ & AX'L(\xi) \leq b(\xi) \;\; \mathbb{P}_{\xi}\text{-a.s.} \end{array} \qquad (\mathcal{NUB})$$

$$x'(\xi') = X'\xi'$$

$$\begin{array}{ll} \min & \mathbb{E}_{\xi'}\left(c(R\xi')^{\top} X'\xi'\right) \\ \text{s.t.} & X' \in \mathbb{R}^{n \times k'} \\ & AX'\xi' \leq b(R\xi') \;\; \mathbb{P}_{\xi'}\text{-a.s.} \end{array} \qquad (\mathcal{LUB})$$

Theorem

- *Problems $\mathcal{NUB}$ and $\mathcal{LUB}$ are equivalent*
- *Problems $\mathcal{NLB}$ and $\mathcal{LLB}$ are equivalent*



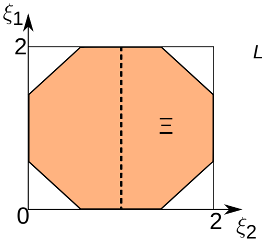
Tractability of $\mathcal{LUB}$

- To apply LDRs to an instance of $\mathcal{SP}$,

$$\Xi := \text{Support}\left(\mathbb{P}_\xi\right) \text{ must be a convex polytope}$$

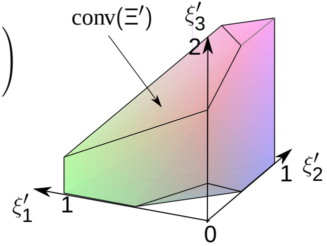
- $\Xi' := \text{Support}\left(\mathbb{P}_{\xi'}\right) = L(\Xi)$ non-convex for non-linear L

$$\implies \text{Find the convex hull of } \Xi'$$



$$L\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}\xi_1\\ \min\{\xi_2, 1\}\\ \max\{\xi_2, 0\}\end{pmatrix}$$

$$R\begin{pmatrix}\xi'_1\\ \xi'_2\\ \xi'_3\end{pmatrix} = \begin{pmatrix}\xi'_1\\ \xi'_1 + \xi'_2\end{pmatrix}$$



Theorem

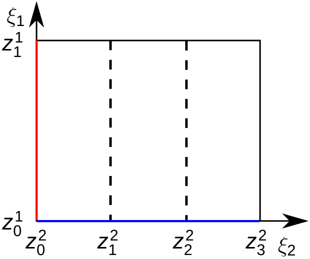
- *$\mathcal{LUB}$ generally intractable*
- *$\mathcal{LLB}$ generally intractable*

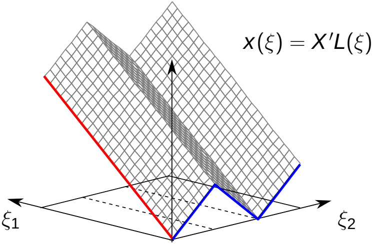
Liftings with Axial Segmentation

$$L_{ij}^{\perp}(\xi) := \begin{cases} \xi_i & \text{if } r_i = 1, \\ \min\left\{\xi_i, z_1^i\right\} & \text{if } r_i > 1,\ j = 1, \\ \max\left\{\min\left\{\xi_i, z_j^i\right\} - z_{j-1}^i, 0\right\} & \text{if } r_i > 1,\ j = 2, \ldots, r_i - 1, \\ \max\left\{\xi_i - z_{j-1}^i, 0\right\} & \text{if } r_i > 1,\ j = r_i. \end{cases}$$

$$R_i^{\perp}(\xi') := \sum_{j=1}^{r_i} \xi'_{ij}.$$

- Lifting depends only on individual ξ_i's

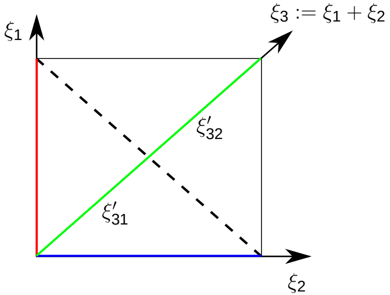


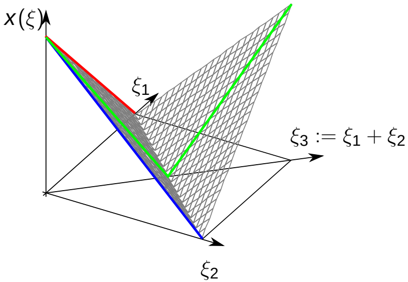


Liftings with General Segmentation

$$L(\xi) := G \circ L^{\perp} \circ F(\xi)$$
$$R(\xi') := F^{+} \circ R^{\perp} \circ G^{+}(\xi')$$

- Piecewise linear w.r.t. linear combinations of ξ_i's

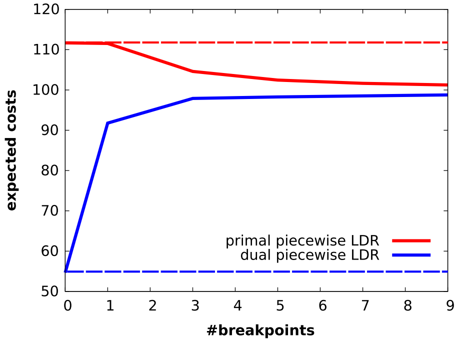




Piecewise Linear Decision Rules

Example problem: capacity expansion of a power grid

- 10 regions with uncertain demand
- 5 power plants with known capacity, uncertain operating costs
- 24 transmission lines with known capacity
- **goal:** meet demand at lowest expected costs, via
  - capacity expansion plan (here-and-now)
  - plant operating policies (wait-and-see)



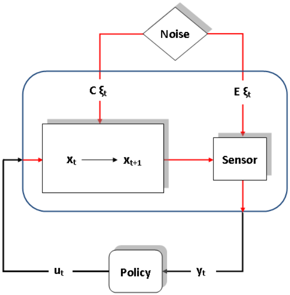
Stochastic Optimal Control

System dynamics:

$$\left.\begin{aligned} \mathbf{x}_{t+1} &= A_t \mathbf{x}_t + B_t \mathbf{u}_t + C_t \boldsymbol{\xi}_t \\ \mathbf{y}_t &= D_t \mathbf{x}_t + E_t \boldsymbol{\xi}_t \end{aligned}\right\} \; t = 1, \ldots, T-1$$

$$\Updownarrow$$

$$\begin{aligned} \mathbf{x} &= B\mathbf{u} + C\boldsymbol{\xi} \\ \mathbf{y} &= D\mathbf{x} + E\boldsymbol{\xi} \end{aligned}$$



Control problem:

$$\begin{aligned} \underset{\mathbf{u},\mathbf{x},\mathbf{s}}{\text{minimize}} \quad & \mathbb{E}\left[\mathbf{u}^\top J_u \mathbf{u} + \mathbf{x}^\top J_x \mathbf{x}\right] \\ \text{subject to} \quad & \left.\begin{aligned} & \mathbf{x} = B\mathbf{u} + C\boldsymbol{\xi} \\ & F_u \mathbf{u} + F_x \mathbf{x} + F_s \mathbf{s} = h \\ & \mathbf{s} \geq 0 \end{aligned}\right\} \; \mathbb{P}\text{-a.s.} \end{aligned}$$

Causal controllers:

$$\mathbf{u}_t = \varphi_t(\mathbf{y}_1, \ldots, \mathbf{y}_t)$$

Controller Information Structure[4]

Purified observations:

- $\boldsymbol{\eta}_t = \boldsymbol{y}_t - \bar{\boldsymbol{y}}_t$
- $\bar{\boldsymbol{y}}_t$ = obervation of noise-free system at time t
- $\boldsymbol{\eta} = G\boldsymbol{\xi}$ where $G = DC + E$

Adapted controllers:

controllers adapted to $\boldsymbol{y}$ $\simeq$ controllers adapted to $\boldsymbol{\eta}$

$\Rightarrow$ Information structure is not decision-dependent

Affine controllers:

controllers affine in $\boldsymbol{y}$ $\simeq$ controllers affine in $\boldsymbol{\eta}$

$\Rightarrow$ Optimising over affine controllers is tractable

[4] Ben-Tal *et al.*, Math. Programming, 2006; Goulart & Kerrigan, Int. J. Control, 2007

Primal Affine Controllers

Controllers affine in y:

$$\begin{aligned}\underset{u,x,s,U}{\text{minimize}} \quad & \mathbb{E}\left[u^\top J_u u + x^\top J_x x\right] \\ \text{subject to} \quad & \left.\begin{array}{l} x = Bu + C\xi, \ \boxed{u = Uy} \\ F_u u + F_x x + F_s s = h \\ s \geq 0 \end{array}\right\} \mathbb{P}\text{-a.s.}\end{aligned}$$

- U is lower block triangular (causal)
- u, x become non-linear (rational) functions of U

Controllers affine in η:

$$\begin{aligned}\underset{u,x,s,Q}{\text{minimize}} \quad & \mathbb{E}\left[u^\top J_u u + x^\top J_x x\right] \\ \text{subject to} \quad & \left.\begin{array}{l} x = Bu + C\xi, \ \boxed{u = Q\eta} \\ F_u u + F_x x + F_s s = h \\ s \geq 0 \end{array}\right\} \mathbb{P}\text{-a.s.}\end{aligned}$$

- Q is lower block triangular (causal)
- u, x become linear functions of Q

Tractable conic program:

$$\begin{aligned}\underset{Q,S}{\text{minimize}} \quad & \text{tr}\left(G^\top Q^\top (J_u + B^\top J_x B) Q G M + 2C^\top J_x B Q G M + C^\top J_x C M\right) \\ \text{subject to} \quad & (F_u + F_x B) Q G + F_x C + F_s S - h e_0^\top = 0 \\ & S \in (\text{cone}(\Xi)^*)^m\end{aligned}$$

Dual Affine Controllers[5]

Dual controllers affine in ξ:

$$\begin{aligned}
\underset{\boldsymbol{u},\boldsymbol{x},\boldsymbol{s}}{\text{minimize}} \quad & \sup_{Y} \mathbb{E}\left[\boldsymbol{u}^\top J_u \boldsymbol{u} + \boldsymbol{x}^\top J_x \boldsymbol{x} + \boldsymbol{\xi}^\top Y^\top (F_u \boldsymbol{u} + F_x \boldsymbol{x} + F_s \boldsymbol{s} - h)\right] \\
\text{subject to} \quad & \left.\begin{array}{l} \boldsymbol{x} = B\boldsymbol{u} + C\boldsymbol{\xi} \\ \boldsymbol{s} \geq 0 \end{array}\right\} \mathbb{P}\text{-a.s.}
\end{aligned}$$

Constraint aggregation:

$$\begin{aligned}
\underset{\boldsymbol{u},\boldsymbol{x},\boldsymbol{s}}{\text{minimize}} \quad & \mathbb{E}\left[\boldsymbol{u}^\top J_u \boldsymbol{u} + \boldsymbol{x}^\top J_x \boldsymbol{x}\right] \\
\text{subject to} \quad & \boldsymbol{x} = B\boldsymbol{u} + C\boldsymbol{\xi} \ \mathbb{P}\text{-a.s.} \\
& \mathbb{E}\left[(F_u \boldsymbol{u} + F_x \boldsymbol{x} + F_s \boldsymbol{s} - h)\boldsymbol{\xi}^\top\right] = 0 \\
& \boldsymbol{s} \geq 0 \ \mathbb{P}\text{-a.s.}
\end{aligned}$$

Tractable conic program:

$$\begin{aligned}
\underset{Q,S}{\text{minimize}} \quad & \operatorname{tr}\left(G^\top Q^\top (J_u + B^\top J_x B) Q G M + 2C^\top J_x B Q G M + C^\top J_x C M\right) \\
\text{subject to} \quad & (F_u + F_x B) Q G + F_x C + F_s S - h e_0^\top = 0 \\
& S M \in \text{cone}(\Xi)^m
\end{aligned}$$

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[5] Hadjiyiannis *et al.*, Optimization Online, 2010

Double Integrator

Assumptions:

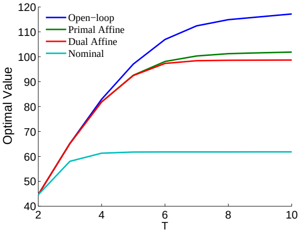
- System dynamics

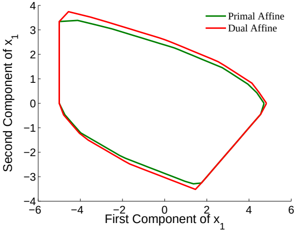
$$x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} u_t + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \xi_t$$

- Perfect state measurements: $y_t = x_t$
- i.i.d. noise $\xi_t \sim \mathcal{U}([0,2])$

Control problem

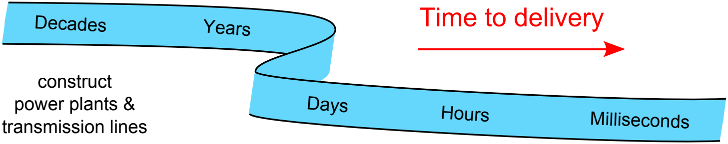
$$\left.\begin{aligned} \underset{u,x}{\text{minimize}} \quad & \mathbb{E}\left[u^\top u + x^\top x\right] \\ \text{subject to} \quad & -(5,5)^\top \le x_t \le (5,5)^\top \\ & (1,1)\, x_t \le 5 \\ & (1,-1)\, x_t \le 5 \\ \text{subject to} \quad & -1 \le u_t \le 1 \end{aligned}\right\} \mathbb{P}\text{-a.s. } \forall t$$





Capacity Expansion in Power Systems

Multiple time scales:



construct
power plants &
transmission lines

schedule
inflexible plants

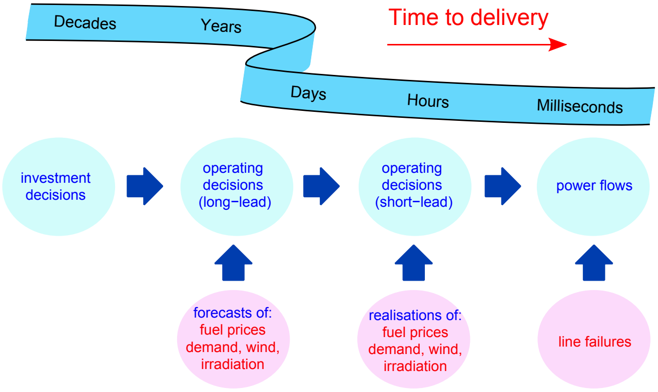
schedule
flexible plants

power flows

Capacity Expansion in Power Systems

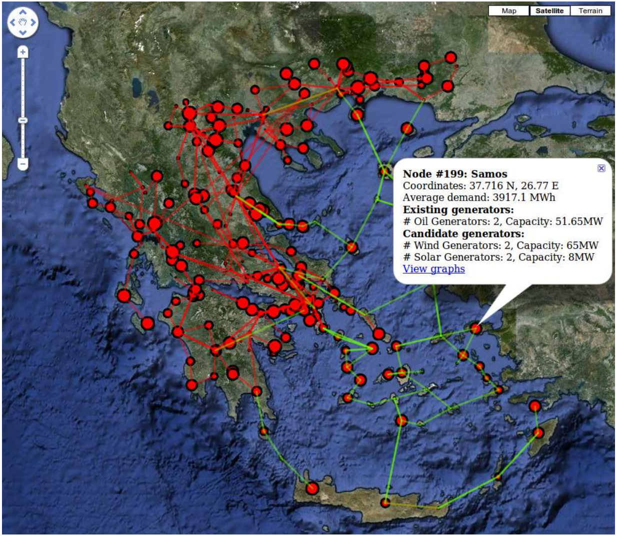
Four-stage stochastic program:

Objective: minimize investment costs + expected operating costs
over next 50 years



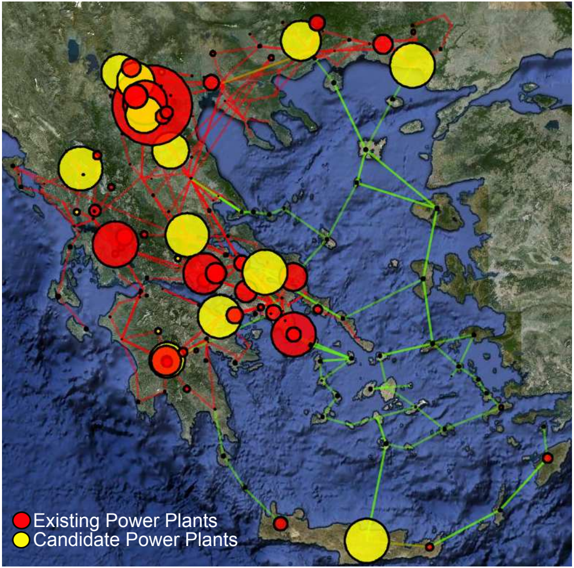
Existing Infrastructure

Power system: 265 buses, 429 transmission lines



Extension Options

Possible additions: 217 power plants, 81 transmission lines



Optimisation Model

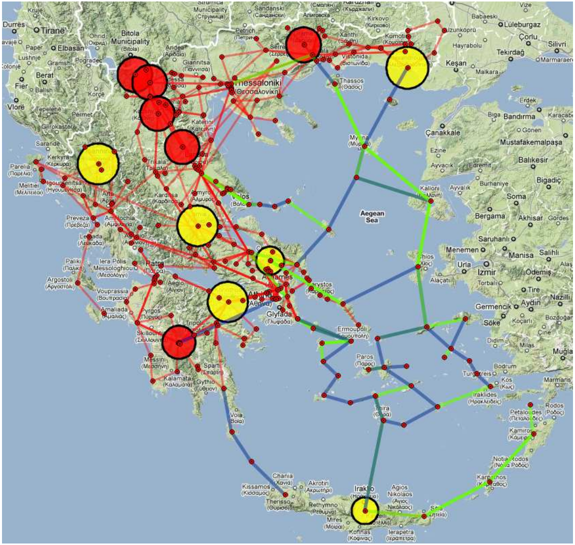
$$\text{minimize} \quad \sum_{n \in N_c} c_n u_n + \sum_{m \in M_c} d_m v_m + \mathbb{E}\left(\sum_{n \in N} \gamma_n g_n \right)$$

subject to

$$\left.\begin{array}{ll}
g_n \quad \mathcal{F}_l\text{-measurable} & \forall n \in N_l \\
g_n \quad \mathcal{F}_s\text{-measurable} & \forall n \in N_s \\
f_m \quad \mathcal{F}\text{-measurable} & \forall m \in M \\
u_n \in \{0,1\},\ v_m \in \{0,1\} & \forall n \in N,\ \forall m \in M \\
u_n = 1 & \forall n \in N_e \\
v_m = 1 & \forall m \in M_e \\
0 \leq g_n \leq \overline{g}_n u_n & \forall n \in N \\
g_n \leq \zeta_n & \forall n \in N_r \\
|f_m| \leq \varphi_m \overline{f}_m v_m & \forall m \in M \\
\sum_{n \in N(k)} g_n - \sum_{m \in M_-(k)} f_m + \sum_{m \in M_+(k)} f_m \geq \delta_k & \forall k \in K
\end{array}\right\} \mathbb{P}\text{-a.s.}$$

Results

Solution time: 2.5 hours



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