Real Options and Game Theory: Introduction and Applications

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Seminar Outline

- Mathematical Background (Dixit and Pindyck, 1994: chs. 3–4)
- Investment and Operational Timing (Dixit and Pindyck, 1994: chs. 5–6 and McDonald, 2005: ch. 17)
- Strategic Interactions (Huisman and Kort, 1999)
- Capacity Switching (Siddiqui and Takashima, 2011)

Topic Outline

★ Wiener process and GBM

★ Itô's lemma

★ Dynamic programming

Wiener Process

- A Wiener process (or Brownian motion) has the following properties:
  - Markov process
  - Independent increments
  - Changes over any finite time interval are normally distributed with variance that increases linearly in time
- Nice property that past patterns have no forecasting value
- For prices, it makes more sense to assume that changes in their logarithms are normally distributed, i.e., prices are lognormally distributed
- More formally for a Wiener process $\{z(t), t \geq 0\}$:
  - $\Delta z = \epsilon_t \sqrt{\Delta t}$, where $\epsilon_t \sim \mathcal{N}(0, 1)$
  - ϵ_t are serially uncorrelated, i.e., $\mathbb{E}[\epsilon_t \epsilon_s] = 0$ for $t \neq s$

Wiener Process: Properties

- Implications of the two conditions are examined by breaking up the time interval T into n units of length Δt each
  - Change in z over T is $z(s+T) - z(s) = \sum_{i=1}^{n} \epsilon_i \sqrt{\Delta t}$, where the ϵ_i are independent
  - Via the CLT, $z(s+T) - z(s)$ is $\mathcal{N}(0, n\Delta t = T)$
  - Variance of the changes increases linearly in time
- Letting Δt become infinitesimally small implies $dz = \epsilon_t \sqrt{dt}$, where $\epsilon_t \sim \mathcal{N}(0,1)$
- This implies that $\mathbb{E}[dz] = 0$ and $\mathbb{V}(dz) = \mathbb{E}[(dz)^2] = dt$
- Coefficient of correlation between two Wiener processes, $z_1(t)$ and $z_2(t)$: $\mathbb{E}[dz_1 dz_2] = \rho_{12} dt$

Brownian Motion with Drift

- Generalise the Wiener process: $dx = \alpha dt + \sigma dz$, where dz is the increment of the Wiener process, α is the drift parameter, and σ is the variance parameter
  - Over time interval Δt, Δx is normal with mean $\mathbb{E}[\Delta x] = \alpha \Delta t$ and variance $\mathbb{V}(\Delta x) = \sigma^2 \Delta t$
  - Given x_0, it is possible to generate sample paths
  - For example, if $\alpha = 0.2$ and $\sigma = 1.0$, then the discretisation with $\Delta t = \frac{1}{12}$ is $x_t = x_{t-1} + 0.01667 + 0.2887\epsilon_t$ (Figure 3.1)

- Optimal forecast is $\hat{x}_{t+T} = x_t + 0.01667T$ and 66% CI is $x_t + 0.01667T \pm 0.2887\sqrt{T}$ (Figure 3.2)

- Mean of $x_t - x_0$ is αt and its SD is $\sigma\sqrt{t}$, so the trend dominates in the long run

Brownian Motion with Drift: Figures 3.1 and 3.2

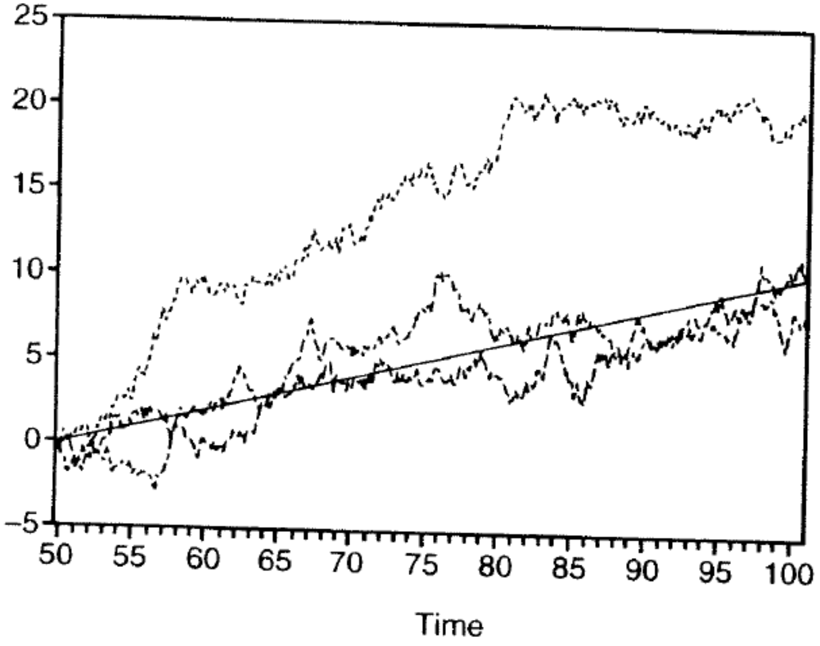

Figure 3.1. Sample Paths of Brownian Motion with Drift

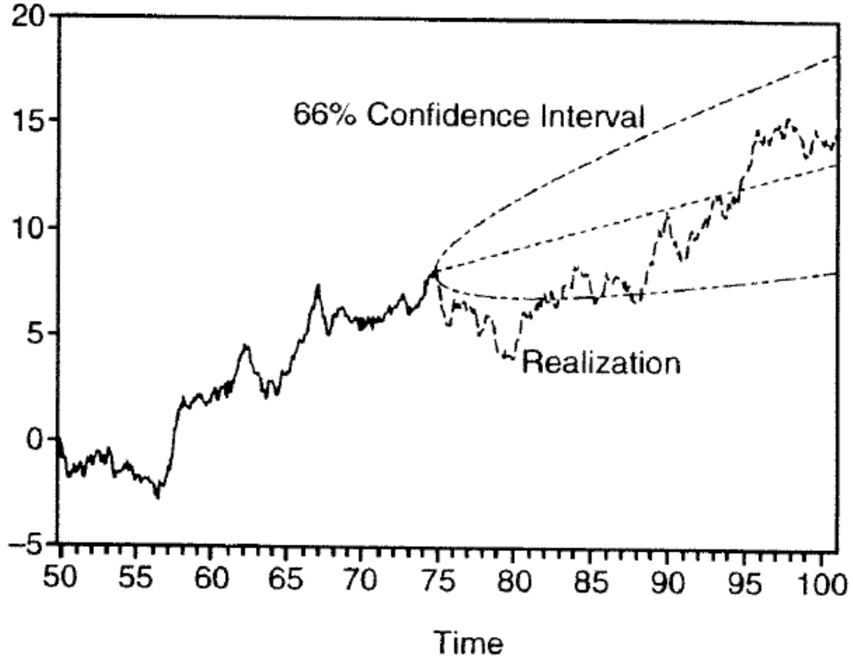

Figure 3.2. Optimal Forecast of Brownian Motion with Drift

Brownian Motion and Random Walks

- Suppose that a discrete-time random walk for which the position is described by variable x makes jumps of $\pm\Delta h$ every Δt time units given the initial position x_0
  - The probability of an upward (downward) jump is p ($q = 1 - p$)
  - Thus, x follows a Markov process with independent increments, i.e., probability distribution of its future position depends only on its current position (Figure 3.3)
- Mean: $\mathbb{E}[\Delta x] = (p - q)\Delta h$; second moment: $\mathbb{E}[(\Delta x)^2] = p(\Delta h)^2 + q(\Delta h)^2 = (\Delta h)^2$; variance: $\mathbb{V}(\Delta x) = (\Delta h)^2[1 - (p - q)^2] = [1 - (2p - 1)^2](\Delta h)^2 = 4pq(\Delta h)^2$
- Thus, if t has $n = \frac{t}{\Delta t}$ steps, then $x_t - x_0$ is a binomial RV with mean $n\mathbb{E}[\Delta x] = \frac{t(p-q)\Delta h}{\Delta t}$ and variance $n\mathbb{V}(\Delta x) = \frac{4pqt(\Delta h)^2}{\Delta t}$

Brownian Motion and Random Walks: Figure 3.3

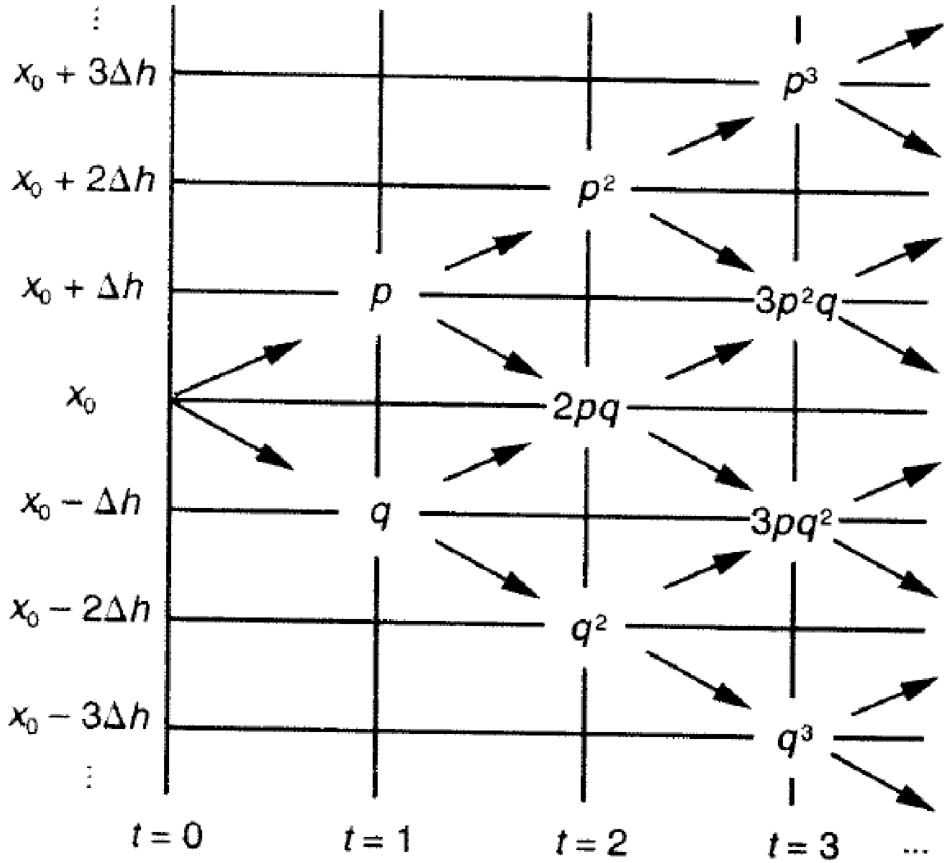

Figure 3.3. *Random Walk Representation of Brownian Motion*

Brownian Motion and Random Walks: Properties

- Choose Δh, Δt, p, and q so that the random walk converges to a Brownian motion as $\Delta t \to 0$
  - $\Delta h = \sigma\sqrt{\Delta t}$
  - $p = \frac{1}{2}\left[1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}\right]$, $q = \frac{1}{2}\left[1 - \frac{\alpha}{\sigma}\sqrt{\Delta t}\right]$
  - Thus, $p - q = \frac{\alpha}{\sigma}\sqrt{\Delta t} = \frac{\alpha}{\sigma^2}\Delta h$
- Substitute these into the formulas for the mean and variance $x_t - x_0$:
  - Mean: $\mathbb{E}[x_t - x_0] = \frac{t\alpha(\Delta h)^2}{\sigma^2 \Delta t} = \frac{t\alpha\sigma^2\Delta t}{\sigma^2\Delta t} = \alpha t$; variance: $\mathbb{V}(x_t - x_0) = \frac{4pqt(\Delta h)^2}{\Delta t} = \frac{4t\sigma^2\Delta t\left[1 - \frac{\alpha^2}{\sigma^2}\Delta t\right]}{4\Delta t} = t\sigma^2\left[1 - \frac{\alpha^2}{\sigma^2}\Delta t\right]$, which goes to $t\sigma^2$ as $\Delta t \to 0$
- Hence, these are the mean and variance of a Brownian motion; furthermore, the binomial distribution approaches the normal one for large n

Generalised Brownian Motion

- An Itô process is $dx = a(x,t)dt + b(x,t)dz$, where dz is the increment of a Wiener process, and both $a(x,t)$ and $b(x,t)$ are known but may be functions of both x and t
  - Mean: $\mathbb{E}[dx] = a(x,t)dt$; second moment: $\mathbb{E}[(dx)^2] = \mathbb{E}[a^2(x,t)(dt)^2 + b^2(x,t)(dz)^2 + 2a(x,t)b(x,t)dtdz] = b^2(x,t)dt$; variance: $\mathbb{V}(dx) = \mathbb{E}[(dx)^2] - (\mathbb{E}[dx])^2 = b^2(x,t)dt$
- A geometric Brownian motion (GBM) has $a(x,t) = \alpha x$ and $b(x,t) = \sigma x$, which implies $dx = \alpha x dt + \sigma x dz$
  - Percentage changes in x are normally distributed, or absolute changes in x are lognormally distributed
  - If $\{y(t), t \geq 0\}$ is a BM with parameters $\left(\alpha - \frac{1}{2}\sigma^2\right)t$ and $\sigma^2 t$, then $\{x(t) \equiv x_0 e^{y(t)}, t \geq 0\}$ is a GBM
  - $m_y(s) = \mathbb{E}[e^{sy(t)}] = e^{s\alpha t - \frac{s\sigma^2 t}{2} + \frac{s^2\sigma^2 t}{2}}$, which implies $\mathbb{E}[y(t)] = \left(\alpha - \frac{1}{2}\sigma^2\right)t$ and $\mathbb{V}(y(t)) = \sigma^2 t$
  - Thus, $\mathbb{E}_{x_0}[x(t)] = \mathbb{E}_{x_0}[x_0 e^{y(t)}] = x_0 m_y(1) = x_0 e^{\alpha t}$ and $\mathbb{V}_{x_0}(x(t)) = \mathbb{E}_{x_0}[(x(t))^2] - (\mathbb{E}_{x_0}[x(t)])^2 = x_0^2 \mathbb{E}_{x_0}[e^{2y(t)}] - x_0^2 e^{2\alpha t} = x_0^2 e^{2\alpha t}[e^{\sigma^2 t} - 1]$

GBM Trajectories

- Expected PV of a GBM assuming discount rate $r > \alpha$ is $\mathbb{E}_{x_0}\left[\int_0^\infty x(t)e^{-rt}dt\right] = \int_0^\infty \mathbb{E}_{x_0}[x(t)]e^{-rt}dt = \int_0^\infty x_0 e^{\alpha t}e^{-rt}dt = \frac{x_0}{r-\alpha}$

- Generate sample paths for $\alpha = 0.09$ and $\sigma = 0.2$ per annum using $x_{1950} = 100$ and one-month intervals, i.e., $x_t - x_{t-1} = 0.0075x_{t-1} + 0.0577x_{t-1}\epsilon_t$, where $\epsilon_t \sim \mathcal{N}(0,1)$ (Figure 3.4)
  - Trend line is obtained by setting $\epsilon_t = 0$
  - Optimal forecast given x_{1974} is $\hat{x}_{1974+T} = (1.0075)^T x_{1974}$, while the CI is $(1.0075)^T(1.0577)^{\pm\sqrt{T}}x_{1974}$ (Figure 3.5)

GBM Trajectories: Figures 3.4 and 3.5

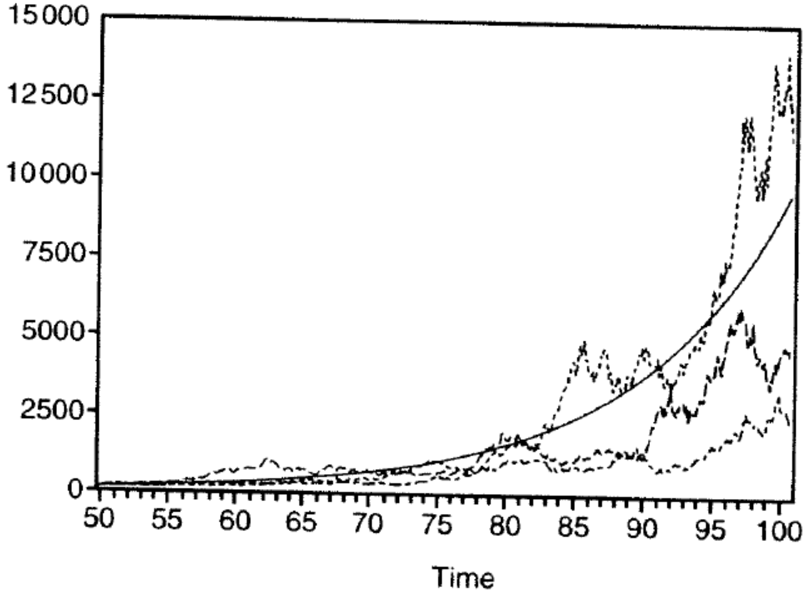

Figure 3.4. Sample Paths of Geometric Brownian Motion

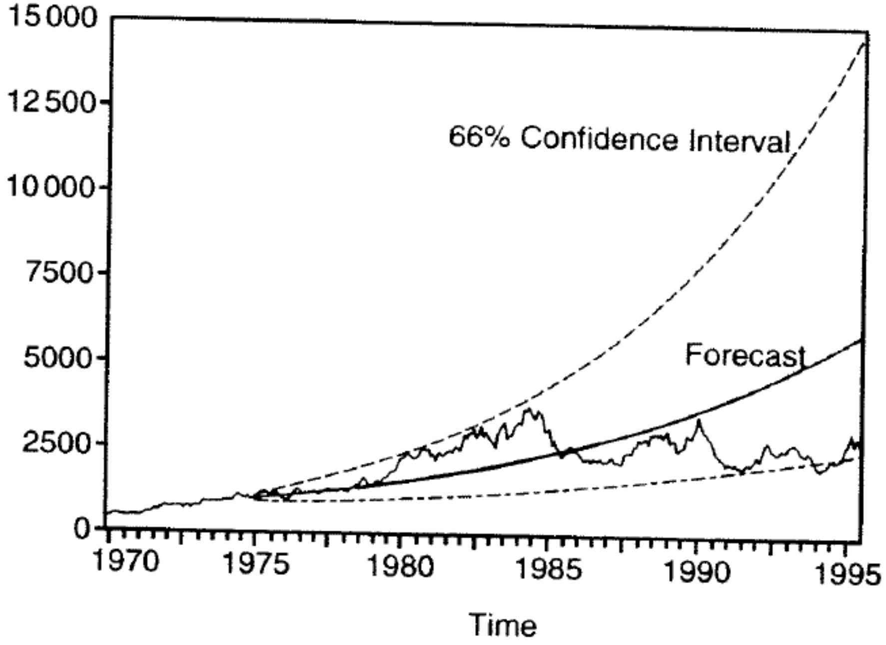

Figure 3.5. Optimal Forecast of Geometric Brownian Motion

Itô's Lemma

- Itô's lemma allows us to integrate and differentiate functions of Itô processes
  - Recall Taylor series expansion for $F(x,t)$: $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2 + \frac{1}{6}\frac{\partial^3 F}{\partial x^3}(dx)^3 + \cdots$
  - Usually, higher-order terms vanish, but here $(dx)^2 = b^2(x,t)dt$ (once terms in $(dt)^{\frac{3}{2}}$ and $(dt)^2$ are ignored), which is linear in dt
  - Thus, $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2 \Rightarrow dF = \left[\frac{\partial F}{\partial t} + a(x,t)\frac{\partial F}{\partial x} + \frac{1}{2}b^2(x,t)\frac{\partial^2 F}{\partial x^2}\right]dt + b(x,t)\frac{\partial F}{\partial x}dz$
  - Intuitively, even if $a(x,t) = 0$ and $\frac{\partial F}{\partial t} = 0$, then $\mathbb{E}[dx] = 0$, but $\mathbb{E}[dF] \neq 0$ because of Jensen's inequality

- Generalise to m Itô processes with $dx_i = a_i(x_1, \ldots, x_m, t)dt + b_i(x_1, \ldots, x_m, t)dz_i$ and $\mathbb{E}[dz_i dz_j] = \rho_{ij}dt$: $dF = \frac{\partial F}{\partial t}dt + \sum_i \frac{\partial F}{\partial x_i}dx_i + \frac{1}{2}\sum_i \sum_j \frac{\partial^2 F}{\partial x_i \partial x_j}dx_i dx_j$

Application to GBM

- If $dx = \alpha x dt + \sigma x dz$ and $F(x) = \ln(x)$, then $F(x)$ follows a BM with parameters $\alpha - \frac{1}{2}\sigma^2$ and σ
  - $\frac{\partial F}{\partial t} = 0$, $\frac{\partial F}{\partial x} = \frac{1}{x}$, $\frac{\partial^2 F}{\partial x^2} = -\frac{1}{x^2}$, which implies that $dF = \frac{dx}{x} - \frac{1}{2x^2}(dx)^2 = \alpha dt + \sigma dz - \frac{1}{2}\sigma^2 dt = (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dz$
- Consider $F(x, y) = xy$ and $G = \ln F$ with $dx = \alpha_x x dt + \sigma_x x dz_x$, $dy = \alpha_y y dt + \sigma_y y dz_y$, and $\mathbb{E}[dz_x dz_y] = \rho dt$
  - $\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2} = 0$ and $\frac{\partial^2 F}{\partial x \partial y} = 1$, which implies $dF = ydx + xdy + dxdy$
  - Substitute dx and dy: $dF = \alpha_x xydt + \sigma_x xydz_x + \alpha_y xydt + \sigma_y xydz_y + xy\sigma_x\sigma_y\rho dt \Rightarrow dF = (\alpha_x + \alpha_y + \rho\sigma_x\sigma_y)Fdt + (\sigma_x dz_x + \sigma_y dz_y)F$, i.e., F is also a GBM
  - Meanwhile, $dG = (\alpha_x + \alpha_y - \frac{1}{2}\sigma_x^2 - \frac{1}{2}\sigma_y^2)dt + \sigma_x dz_x + \sigma_y dz_y$
- Discounted PV: $F(x) = x^\theta$ and x follows a GBM
  - F follows a GBM, too: $dF = \theta x^{\theta-1}dx + \frac{1}{2}\theta(\theta - 1)x^{\theta-2}(dx)^2 = F[\theta\alpha + \frac{1}{2}\theta(\theta-1)\sigma^2]dt + \theta\sigma F dz \Rightarrow \mathbb{E}_{x_0}[F(x(t))] = F(x_0)e^{t(\theta\alpha + \frac{1}{2}\theta(\theta-1)\sigma^2)}$
  - Thus, $\mathbb{E}_{x_0}\left[\int_0^\infty F(x(t))e^{-rt}dt\right] = \frac{x_0^\theta}{r - \alpha\theta - \frac{1}{2}\theta(\theta-1)\sigma^2}$

Stochastic Discount Factor

- Proposition: The conditional expectation of the stochastic discount factor, $\mathbb{E}_p\left[e^{-\rho\tau}\right]$, is the power function, $\left(\frac{p}{P^*}\right)^{\beta_1}$, where $\tau \equiv \min\left\{t : P_t \geq P^*\right\}$, $dP = \alpha P dt + \sigma P dz$, and $P_0 \equiv p$.
- Proof: Let $g(p) \equiv \mathbb{E}_p\left[e^{-\rho\tau}\right]$
  - $g(p) = o(dt)e^{-\rho dt} + (1 - o(dt))e^{-\rho dt}\mathbb{E}_p\left[g(p + dP)\right]$
  - $\Rightarrow g(p) = o(dt)e^{-\rho dt} + (1 - o(dt))e^{-\rho dt}\mathbb{E}_p\left[g(p) + dPg'(p) + \frac{1}{2}(dP)^2 g''(p) + o(dt)\right]$
  - $\Rightarrow g(p) = o(dt) + e^{-\rho dt}g(p) + e^{-\rho dt}\alpha p g'(p)dt + e^{-\rho dt}\frac{1}{2}\sigma^2 p^2 g''(p)dt$
  - $\Rightarrow g(p) = o(dt) + (1 - \rho dt)g(p) + (1 - \rho dt)\alpha p g'(p)dt + (1 - \rho dt)\frac{1}{2}\sigma^2 p^2 g''(p)dt$
  - $\Rightarrow -\rho g(p) + \alpha p g'(p) + \frac{1}{2}\sigma^2 p^2 g''(p) = \frac{o(dt)}{dt}$
  - $\Rightarrow g(p) = a_1 p^{\beta_1} + a_2 p^{\beta_2}$
  - $\lim_{p\to 0} g(p) = 0 \Rightarrow a_2 = 0$ and $g(P^*) = 1 \Rightarrow a_1 = \frac{1}{P^{*\beta_1}}$

Dynamic Programming: Many-Period Example

- Now, let the state variable x_t be continuous and the control variable u_t represent the possible choices made at time t
  - Let the immediate profit flow be $\pi_t(x_t, u_t)$ and $\Phi_t(x_{t+1}|x_t, u_t)$ be the CDF of the state variable next period given current information
  - Given the discount rate ρ and the Bellman Principle of Optimality, the expected NPV of the cash flows to go from period t is $F_t(x_t) = \max_{u_t}\left\{\pi_t(x_t, u_t) + \frac{1}{(1+\rho)}\mathbb{E}_t[F_{t+1}(x_{t+1})]\right\}$
  - Use the termination value at time T and work backwards to solve for successive values of u_t: $F_{T-1}(x_{T-1}) = \max_{u_{T-1}}\left\{\pi_{T-1}(x_{T-1}, u_{T-1}) + \frac{1}{(1+\rho)}\mathbb{E}_{T-1}[\Omega_T(x_T)]\right\}$
- With an infinite horizon, it is possible to solve the problem recursively due to independence from time and the downward scaling due to the discount factor: $F(x) = \max_u\left\{\pi(x, u) + \frac{1}{(1+\rho)}\mathbb{E}[F(x')|x, u]\right\}$

Dynamic Programming: Optimal Stopping

- Suppose that the choice is binary: either continue (to wait or to produce) or to terminate (waiting or production)
  - Bellman equation is now $\max\left\{\Omega(x), \pi(x) + \frac{1}{(1+\rho)}\mathbb{E}[F(x')|x]\right\}$
  - Focus on case where it is optimal to continue for $x > x^*$ and stop otherwise
  - Continuation is more attractive for higher x if: (i) immediate profit from continuation becomes larger relative to the termination payoff, i.e., $\pi(x) + \frac{1}{(1+\rho)}\mathbb{E}[\Omega(x')|x] - \Omega(x)$ is increasing in x, and (ii) current advantage should not be likely to be reversed in the near future, i.e., require first-order stochastic dominance
  - Both conditions are satisfied in the applications studied here: (i) always holds, and (ii) is true for random walks, Brownian motion, MR processes, and most other economic applications
  - In general, may have stopping threshold that varies with time, $x^*(t)$

Dynamic Programming: Continuous Time

★ In continuous time, the length of the time period, Δt, goes to zero and all cash flows are expressed in terms of rates

- Bellman equation is now $F(x,t) = \max_u \left\{ \pi(x,u,t)\Delta t + \frac{1}{(1+\rho\Delta t)} \mathbb{E}[F(x',t+\Delta t)|x,u] \right\}$
- Multiply by $(1+\rho\Delta t)$ and re-arrange: $\rho\Delta t F(x,t) = \max_u \{\pi(x,u,t)\Delta t(1+\rho\Delta t) + \mathbb{E}[F(x',t+\Delta t) - F(x,t)|x,u]\} = \max_u \{\pi(x,u,t)\Delta t(1+\rho\Delta t) + \mathbb{E}[\Delta F|x,u]\}$
- Divide by Δt and let it go to zero to obtain $\rho F(x,t) = \max_u \left\{ \pi(x,u,t) + \frac{\mathbb{E}[dF|x,u]}{dt} \right\}$
- Intuitively, the instantaneous rate of return on the asset must equal its expected net appreciation

Dynamic Programming: Itô Processes

- Suppose that $dx = a(x,u,t)dt + b(x,u,t)dz$ and $x' = x + dx$
- Apply Itô's lemma to the value function, F:
  - $\mathbb{E}[F(x+\Delta x, t+\Delta t)|x,u] = F(x,t) + [F_t(x,t) + a(x,u,t)F_x(x,t) + \frac{1}{2}b^2(x,u,t)F_{xx}(x,t)]\Delta t + o(\Delta t)$
  - Return equilibrium condition is now $\rho F(x,t) = \max_u \left\{\pi(x,u,t) + F_t(x,t) + a(x,u,t)F_x(x,t) + \frac{1}{2}b^2(x,u,t)F_{xx}(x,t)\right\}$
  - Next, find optimal u as a function of $F_t(x,t)$, $F_x(x,t)$, $F_{xx}(x,t)$, x, t, and underlying parameters
  - Subsitute it back into the return equilibrium condition to obtain a second-order PDE with F as the dependent variable and x and t as the independent ones
  - Solution procedure is typically to start at the terminal time T and work backwards
- When time horizon is infinite, t drops out of the equation:
  - $\rho F(x) = \max_u \left\{\pi(x,u) + a(x,u)F'(x) + \frac{1}{2}b^2(x,u)F''(x)\right\}$

Dynamic Programming: Optimal Stopping and Smooth Pasting

- Consider a binary decision problem: can either continue to obtain a profit flow (with continuation value) or stop to obtain a termination payoff where $dx = a(x,t)dt + b(x,t)dz$
  - In this case, a threshold policy with $x^*(t)$ exists, and the Bellman equation is $\rho F(x,t)dt = \max\{\Omega(x,t)dt, \pi(x,t)dt + \mathbb{E}[dF|x]\}$
  - The RHS is larger in the continuation region, so applying Itô's lemma gives $\frac{1}{2}b^2(x,t)F_{xx}(x,t)+a(x,t)F_x(x,t)+F_t(x,t)-\rho F(x,t)+\pi(x,t)=0$
  - The PDE can be solved for $F(x,t)$ for $x > x^*(t)$ subject to the boundary condition $F(x^*(t),t) = \Omega(x^*(t),t)\ \forall t$ (value-matching condition)
  - A second condition is necessary to find the free boundary: $F_x(x^*(t),t) = \Omega_x(x^*(t),t)\ \forall t$ (smooth-pasting condition)
  - The latter may be thought of as a first-order necessary condition, i.e., if the two curves met at a kink, then the optimal stopping would occur elsewhere

Dynamic Programming: Optimal Abandonment

- You own a machine that produces profit, x, that evolves according to a BM, i.e., $dx = adt + bdz$, where $a < 0$ to reflect decay of the machine over time

- The lifetime of the machine is T years, discount rate is ρ, and we must find the optimal threshold profit level, $x^*(t)$, below which to abandon the machine (zero salvage value)
  - Corresponding PDE is $\frac{1}{2}b^2 F_{xx}(x,t) + aF_x(x,t) + F_t(x,t) - \rho F(x,t) + x = 0$
  - PDE is solved numerically for $T = 10$, $a = -0.1$, $b = 0.2$, and $\rho = 0.10$ using discrete time steps of $\Delta t = 0.01$
  - Solution in Figure 4.1 indicates that for lifetimes greater than ten years, the optimal abandonment threshold is about -0.17
  - As lifetime is reduced, it becomes easier to abandon the machine

Dynamic Programming Example: Figure 4.1

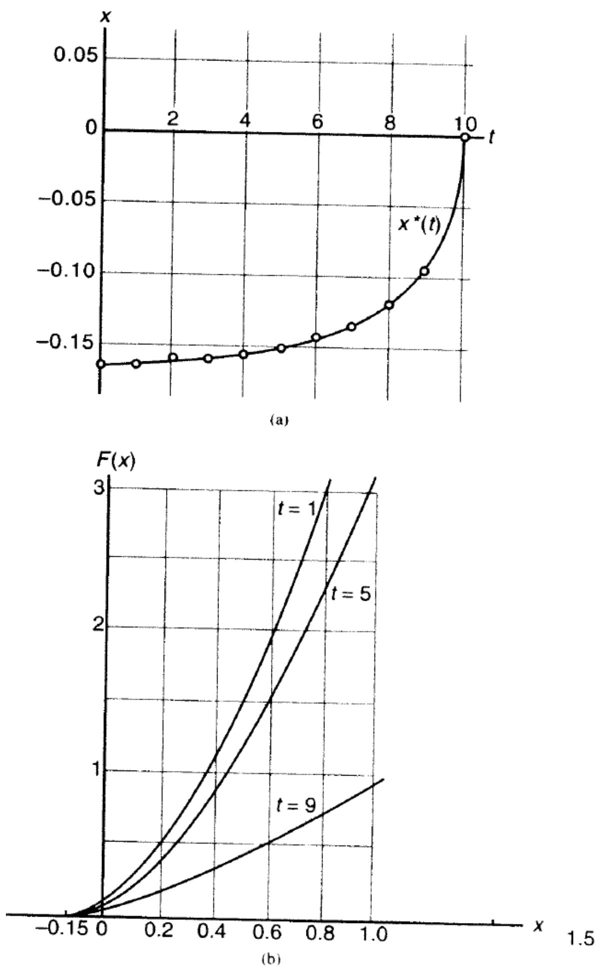

Figure 4.1. Depreciation and Abandonment

Dynamic Programming: Optimal Abandonment

★ Assume an effectively infinite lifetime to obtain an ODE instead of a PDE: $\frac{1}{2}b^2F''(x) + aF'(x) - \rho F(x) + x = 0$

- Homogeneous solution is $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$
- Substituting derivatives into the homogeneous portion of the PDE yields $c_1e^{r_1x}(\frac{1}{2}b^2r_1^2 + ar_1 - \rho) + c_2e^{r_2x}(\frac{1}{2}b^2r_2^2 + ar_2 - \rho) = 0$
- The terms in the parentheses must be equal to zero, i.e., $r_1 = \frac{-a+\sqrt{a^2+2b\rho}}{b^2} = 5.584 > 0$ and $r_2 = \frac{-a-\sqrt{a^2+2b\rho}}{b^2} = -0.854 < 0$
- Particular solution: $Y(x) = Ax + B$, $Y'(x) = A$, and $Y''(x) = 0$
- Substituting these into the original PDE yields $aA - \rho(Ax + B) + x = 0 \Rightarrow A = \frac{1}{\rho}, B = \frac{a}{\rho^2}$
- Thus, $Y(x) = \frac{x}{\rho} + \frac{a}{\rho^2}$, and $F(x) = c_1e^{r_1x} + c_2e^{r_2x} + \frac{x}{\rho} + \frac{a}{\rho^2}$
- Boundary conditions: (i) $F(x^*) = 0$, (ii) $F'(x^*) = 0$, (iii) $\lim_{x\to\infty} F(x) = Y(x)$
- The third one implies that $c_1 = 0$, i.e., $F(x) = c_2e^{r_2x} + \frac{x}{\rho} + \frac{a}{\rho^2}$
- First two conditions imply $x^* = -\frac{a}{\rho} + \frac{1}{r_2} = -0.17$ and $c_2 = -\frac{e^{-r_2x^*}}{r_2\rho}$

Seminar Outline

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- ★ Strategic Interactions (Huisman and Kort, 1999)
- ★ Capacity Switching (Siddiqui and Takashima, 2011)

Topic Outline

★ Basic model and NPV approach

★ Dynamic programming solution

★ Features of optimal investment

★ Embedded options

★ Another approach: optimal stopping time

Basic Model: Optimal Timing

- Suppose project value, V, evolves according to a GBM, i.e., $dV = \alpha V dt + \sigma V dz$, which may be obtained at a sunk cost of I

- When is the optimal time to invest?
  - A perpetual option, i.e., calendar time is not important
  - Ignore temporary suspension or other embedded options
  - Can use both dynamic programming and contingent claims methods

- Problem formulation: $\max_T \mathbb{E}_{V_0}[(V_T - I)e^{-\rho T}]$
  - Assume $\delta \equiv \rho - \alpha > 0$, otherwise it is always better to wait indefinitely

Basic Model: Deterministic Case

- Suppose that $\sigma = 0$, i.e., $V(t) = V_0 e^{\alpha t}$ for $V_0 \equiv V(0)$
  - $F(V) \equiv \max_T e^{-\rho T}(Ve^{\alpha T} - I)$
  - If $\alpha \leq 0$, then $F(V) = \max[V - I, 0]$
  - Otherwise, for $0 < \alpha < \rho$, waiting may be better because either (i) $V < I$ or (ii) $V \geq I$, but discounting of future sunk cost is greater than that in the future project value
  - Thus, the FONC is $\frac{dF(V)}{dT} = 0 \Rightarrow (\rho - \alpha)Ve^{-(\rho-\alpha)T} = \rho I e^{-\rho T} \Rightarrow$ $T^* = \max\left\{\frac{1}{\alpha}\ln\left\{\frac{\rho I}{(\rho-\alpha)V}\right\}, 0\right\}$
  - Reason for delaying is that the MC is depreciating over time by more than the MB
- Substitute T^* to determine $V^* = \frac{\rho I}{(\rho-\alpha)} > I$
- And, $F(V) = \left(\frac{\alpha I}{\rho-\alpha}\right)\left[\frac{(\rho-\alpha)V}{\rho I}\right]^{\frac{\rho}{\alpha}}$ if $V \leq V^*$ ($F(V) = V - I$ otherwise)
- Figure 5.1 indicates that greater α increases V^*

Basic Model: Figure 5.1

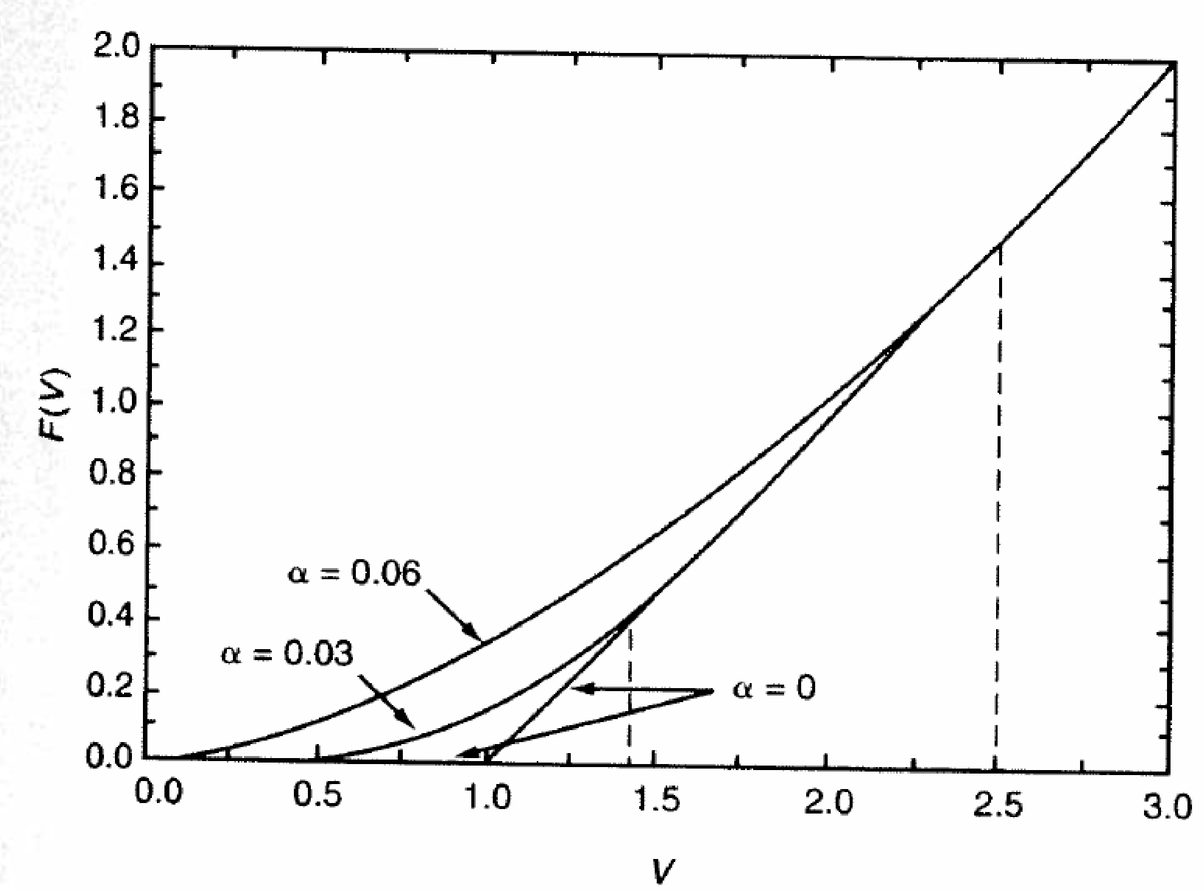


Figure 5.1. *Value of Investment Opportunity, $F(V)$, for $\sigma = 0$, $\rho = 0.1$*

Dynamic Programming Solution

- Bellman equation for continuation is $\rho F dt = \mathbb{E}[dF]$
- Expand the RHS via Itô's lemma: $dF = F'(V)dV + \frac{1}{2}F''(V)(dV)^2 \Rightarrow \mathbb{E}[dF] = F'(V)\alpha V dt + \frac{1}{2}F''(V)\sigma^2 V^2 dt$
- Substitution into the Bellman equation yields the ODE $\frac{1}{2}F''(V)\sigma^2 V^2 + F'(V)\alpha V - \rho F(V) = 0$
  - Equivalently, $\frac{1}{2}F''(V)\sigma^2 V^2 + F'(V)(\rho - \delta)V - \rho F(V) = 0$
  - Three boundary conditions: (i) $F(0) = 0$, (ii) $F(V^*) = V^* - I$, and (iii) $F'(V^*) = 1$
  - General solution to the ODE is $F(V) = A_1 V^{\beta_1} + A_2 V^{\beta_2}$
  - Taking derivatives, we have $F'(V) = A_1\beta_1 V^{\beta_1 - 1} + A_2\beta_2 V^{\beta_2 - 1}$ and $F''(V) = A_1\beta_1(\beta_1 - 1)V^{\beta_1 - 2} + A_2\beta_2(\beta_2 - 1)V^{\beta_2 - 2}$
  - Substitution into the ODE yields $A_1 V^{\beta_1}[\frac{1}{2}\sigma^2\beta_1(\beta_1 - 1) + \beta_1(\rho - \delta) - \rho] + A_2 V^{\beta_2}[\frac{1}{2}\sigma^2\beta_2(\beta_2 - 1) + \beta_2(\rho - \delta) - \rho] = 0$
  - Thus, $\beta_1 = \frac{1}{2} - \frac{(\rho-\delta)}{\sigma^2} + \sqrt{\left[\frac{\rho-\delta}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}$ and $\beta_2 = \frac{1}{2} - \frac{(\rho-\delta)}{\sigma^2} - \sqrt{\left[\frac{\rho-\delta}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}$

Solution Features

- The characteristic quadratic, $\mathcal{Q}(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) + (\rho - \delta)\beta - \rho$, has two roots such that $\beta_1 > 1$ and $\beta_2 < 0$
  - $\mathcal{Q}(\beta)$ has a positive coefficient for β^2, i.e., it is an upward-pointing parabola
  - Note that $\mathcal{Q}(1) = -\delta < 0$, which means that $\beta_1 > 1$
  - $\mathcal{Q}(0) = -\rho$, which means that $\beta_2 < 0$ (Figure 5.2)

- Consequently, the first boundary condition implies that $A_2 = 0$, i.e., $F(V) = A_1 V^{\beta_1}$
  - Using the VM and SP conditions, we obtain $V^* = \frac{\beta_1}{\beta_1 - 1} I$ and $A_1 = \frac{(V^* - I)}{(V^*)^{\beta_1}} = \frac{(\beta_1 - 1)^{\beta_1 - 1}}{[(\beta_1)^{\beta_1} I^{\beta_1 - 1}]}$
  - Since $\beta_1 > 1$, we also have $V^* > I$

Characteristic Quadratic Function: Figure 5.2

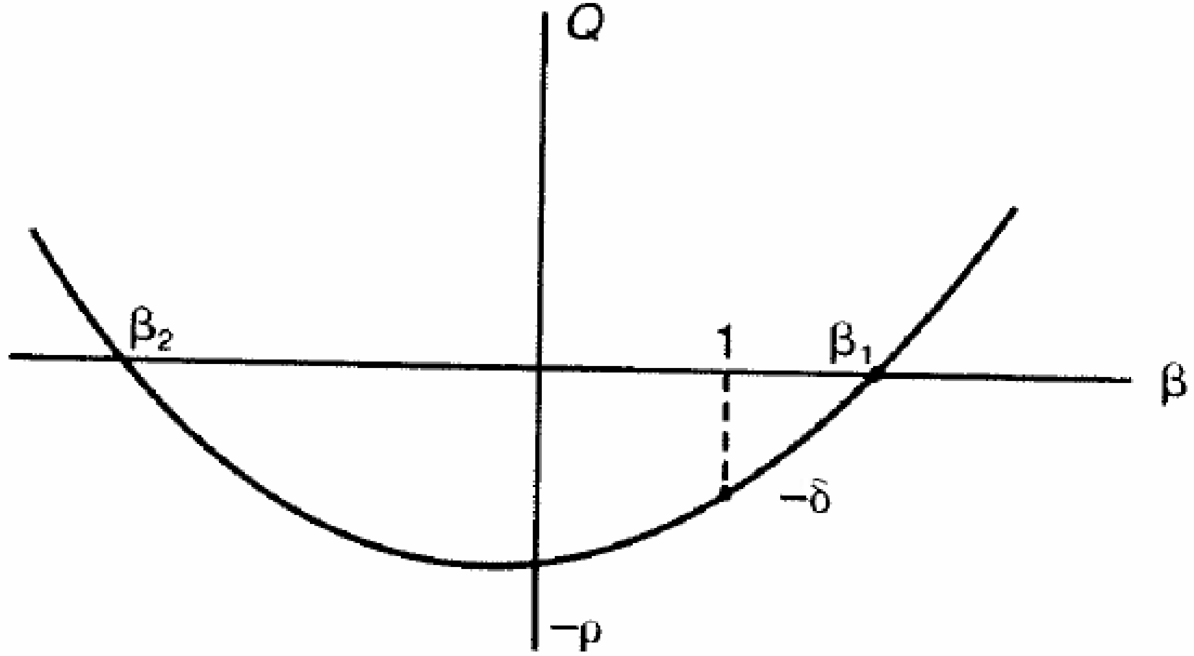


Figure 5.2. The Fundamental Quadratic

Optimal Investment: Comparative Statics

- $\frac{\partial \beta_1}{\partial \sigma} < 0$
  - Differentiate $\mathcal{Q}(\beta)$ totally and evaluate it at β_1
  - $\frac{\partial \mathcal{Q}}{\partial \beta}\frac{\partial \beta_1}{\partial \sigma} + \frac{\partial \mathcal{Q}}{\partial \sigma} = 0 \Rightarrow \frac{\partial \beta_1}{\partial \sigma} = -\frac{\partial \mathcal{Q}/\partial \sigma}{\partial \mathcal{Q}/\partial \beta}$
  - Know that $\frac{\partial \mathcal{Q}}{\partial \beta} > 0$ at β_1 via Figure 5.2 and $\frac{\partial \mathcal{Q}}{\partial \sigma} = \sigma\beta(\beta - 1) > 0$ at $\beta_1 > 1$
  - Thus, $\frac{\partial \beta_1}{\partial \sigma} < 0$ and $\frac{\beta_1}{\beta_1 - 1}$ increases with σ
- Similarly, $\frac{\partial \beta_1}{\partial \delta} = -\frac{\partial \mathcal{Q}/\partial \delta}{\partial \mathcal{Q}/\partial \beta} > 0$
  - For $\beta_1 > 1$, $\frac{\partial \mathcal{Q}}{\partial \delta} = -\beta < -1$
  - Thus, $\frac{\partial \beta_1}{\partial \delta} > 0$ and $\frac{\beta_1}{\beta_1 - 1}$ decreases with δ
- Finally, $\frac{\partial \beta_1}{\partial \rho} = -\frac{\partial \mathcal{Q}/\partial \rho}{\partial \mathcal{Q}/\partial \beta} < 0$
  - For $\beta_1 > 1$, $\frac{\partial \mathcal{Q}}{\partial \rho} = \beta > 1$
  - Thus, $\frac{\partial \beta_1}{\partial \rho} < 0$ and $\frac{\beta_1}{\beta_1 - 1}$ increases with ρ
- As $\sigma \to \infty$, $\beta_1 \to 1$ and $V^* \to \infty$, whereas as $\sigma \to 0$, $\beta_1 \to \frac{\rho}{\rho - \delta}$ and $V^* \to \frac{\rho}{\delta} I$ for $\alpha > 0$

Optimal Investment: Comparison to Neoclassical Theory

- Marshallian analysis is to compare $V_0 \equiv \mathbb{E}_{\pi_0} \int_0^\infty \pi_s e^{-\rho s} ds = \int_0^\infty \mathbb{E}_{\pi_0}[\pi_s] e^{-\rho s} ds = \frac{\pi_0}{\rho - \alpha}$ with I
  - Invest if $V_0 \geq I$ or $\pi_0 \geq (\rho - \alpha) I$
  - Real options approach says to invest when $\pi_0 \geq \pi^* \equiv \frac{\beta_1}{\beta_1 - 1}(\rho - \alpha) I > (\rho - \alpha) I$

- Tobin's q is the ratio of the value of the existing capital goods to the their current reproduction cost
  - Rule is to invest when $q \geq 1$
  - If we interpret q as being $\frac{V}{I}$, then the real options threshold is $q^* = \frac{\beta_1}{\beta_1 - 1} > 1$
  - Hence, the real options definition of q adds option value to the PV of assets in place

Project Value without Operating Costs

★ Suppose that the output price, P, follows a GBM and the firm produces one unit per year forever

- Without operating costs and ruling out speculative bubbles, the value of the project is $V(P) = \mathbb{E}_P \int_0^\infty P_t e^{-\rho t} dt = \int_0^\infty \mathbb{E}_P [P_t] e^{-\rho t} dt = \int_0^\infty P e^{-(\rho-\alpha)t} dt = \frac{P}{\delta}$
- We can now find the value of the option to invest, $F(P)$, which will satisfy the ODE $\frac{1}{2}\sigma^2 P^2 F''(P) + (\rho - \delta)PF'(P) - \rho F(P) = 0$: $F(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2}$
- Boundary condition $F(0) = 0 \Rightarrow A_2 = 0$
- VM and SP conditions imply: (i)$A_1 (P^*)^{\beta_1} = \frac{P^*}{\delta} - I$ and (ii) $\beta_1 A_1 (P^*)^{\beta_1 - 1} = \frac{1}{\delta}$
- Therefore, $P^* = \frac{\beta_1}{\beta_1 - 1} \delta I$ and $A_1 = \frac{(\beta_1 - 1)^{\beta_1 - 1} I^{-(\beta_1 - 1)}}{(\delta \beta_1)^{\beta_1}}$
- Note that $V^* = \frac{P^*}{\delta} = \frac{\beta_1}{\beta_1 - 1} I > I$

Operating Costs and Temporary Suspension: Value of the Project

- Suppose now that the project incurs operating cost, C, but it may be costlessly suspended or resumed once installed
  - Instantaneous profit flow is $\pi(P) = \max[P - C, 0]$, i.e., project owner has infinite embedded operational options
  - Thus, the value of an active project will be worth more than simply the NPV of the cash flows
- Value the project, $V(P)$, via usual dynamic programming approach
  - Unlike the option to invest, we now have a profit flow, $\pi(P)$, which implies that the ODE becomes $\frac{1}{2}\sigma^2 P^2 V''(P) + (\rho - \delta) P V'(P) - \rho V(P) + \pi(P) = 0$
  - For $P < C$, only the homogeneous part of the solution is valid, i.e., $V(P) = K_1 P^{\beta_1} + K_2 P^{\beta_2}$
  - With $P \geq C$, we also have the particular solution $D_1 P + D_2 C + D_3$
  - Substitution into the ODE yields $D_1 = \frac{1}{\delta}, D_2 = -\frac{1}{\rho}, D_3 = 0$
  - Therefore, $V(P) = B_1 P^{\beta_1} + B_2 P^{\beta_2} + \frac{P}{\delta} - \frac{C}{\rho}$ for $P \geq C$

Operating Costs and Temporary Suspension: Value of the Project

- For $P < C$, $V(P)$ represents the option value of resuming a suspended project
  - Intuitively, this must increase in P and be worthless for very small P
  - Only when $K_2 = 0$ does this hold; thus, $V(P) = K_1 P^{\beta_1}$ for $P < C$

- For $P \geq C$, $V(P)$ is the value of an active project inclusive of the option to suspend operations
  - The suspension option is valuable only for small P and becomes worthless for large P
  - Thus, $B_1 = 0$ and $V(P) = B_2 P^{\beta_2} + \frac{P}{\delta} - \frac{C}{\rho}$ for $P \geq C$

- Find K_1 and B_2 via VM and SP at $P = C$
  - $K_1 C^{\beta_1} = B_2 C^{\beta_2} + \frac{C}{\delta} - \frac{C}{\rho}$ and $\beta_1 K_1 C^{\beta_1 - 1} = \beta_2 B_2 C^{\beta_2 - 1} + \frac{1}{\delta}$
  - $K_1 = \frac{C^{1-\beta_1}}{\beta_1 - \beta_2}\left(\frac{\beta_2}{\rho} - \frac{(\beta_2 - 1)}{\delta}\right) > 0$, $B_2 = \frac{C^{1-\beta_2}}{\beta_1 - \beta_2}\left(\frac{\beta_1}{\rho} - \frac{(\beta_1 - 1)}{\delta}\right) > 0$
  - $V(P)$ is increasing (decreasing) in σ (δ) (Figures 6.1 and 6.2)

Operating Costs and Temporary Suspension: Figure 6.1

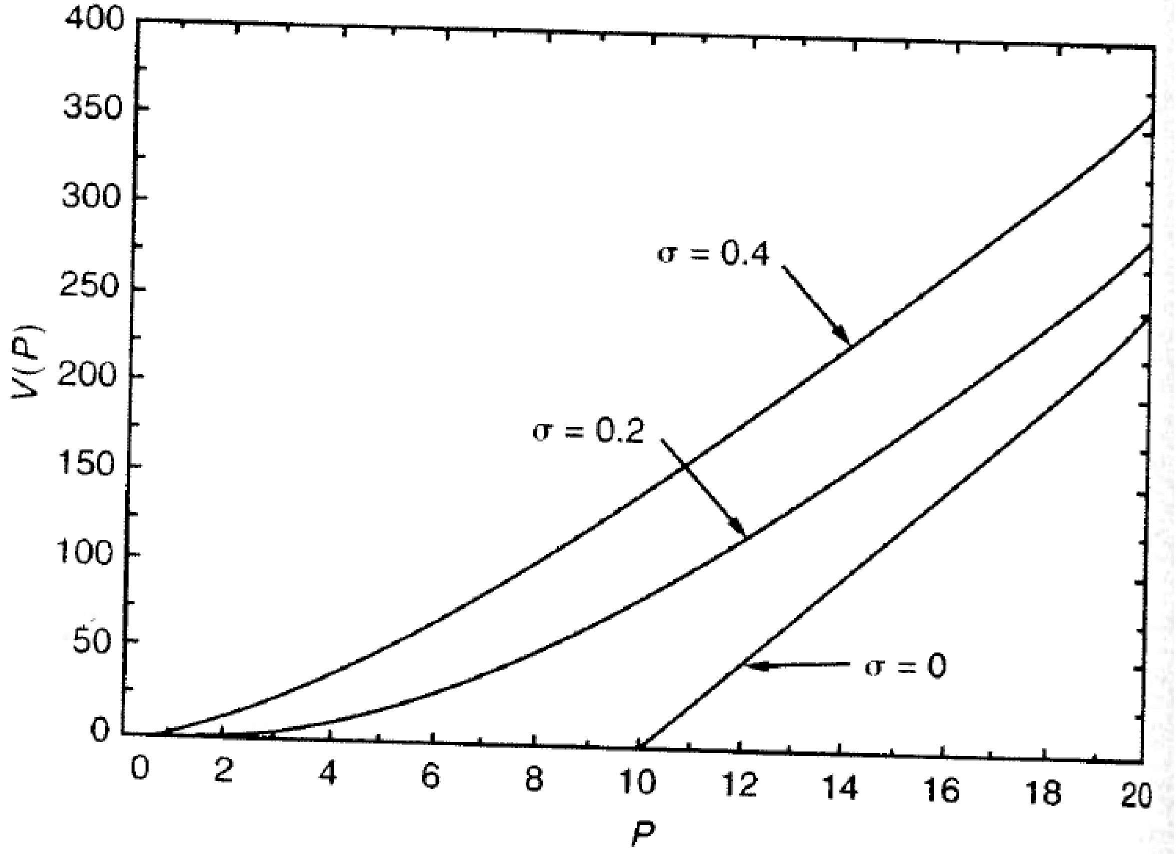


Figure 6.1. *Value of Project, $V(P)$, for $\sigma = 0, 0.2, 0.4$*

(Note: $r = \delta = 0.04$, and $C = 10$)

Operating Costs and Temporary Suspension: Figure 6.2

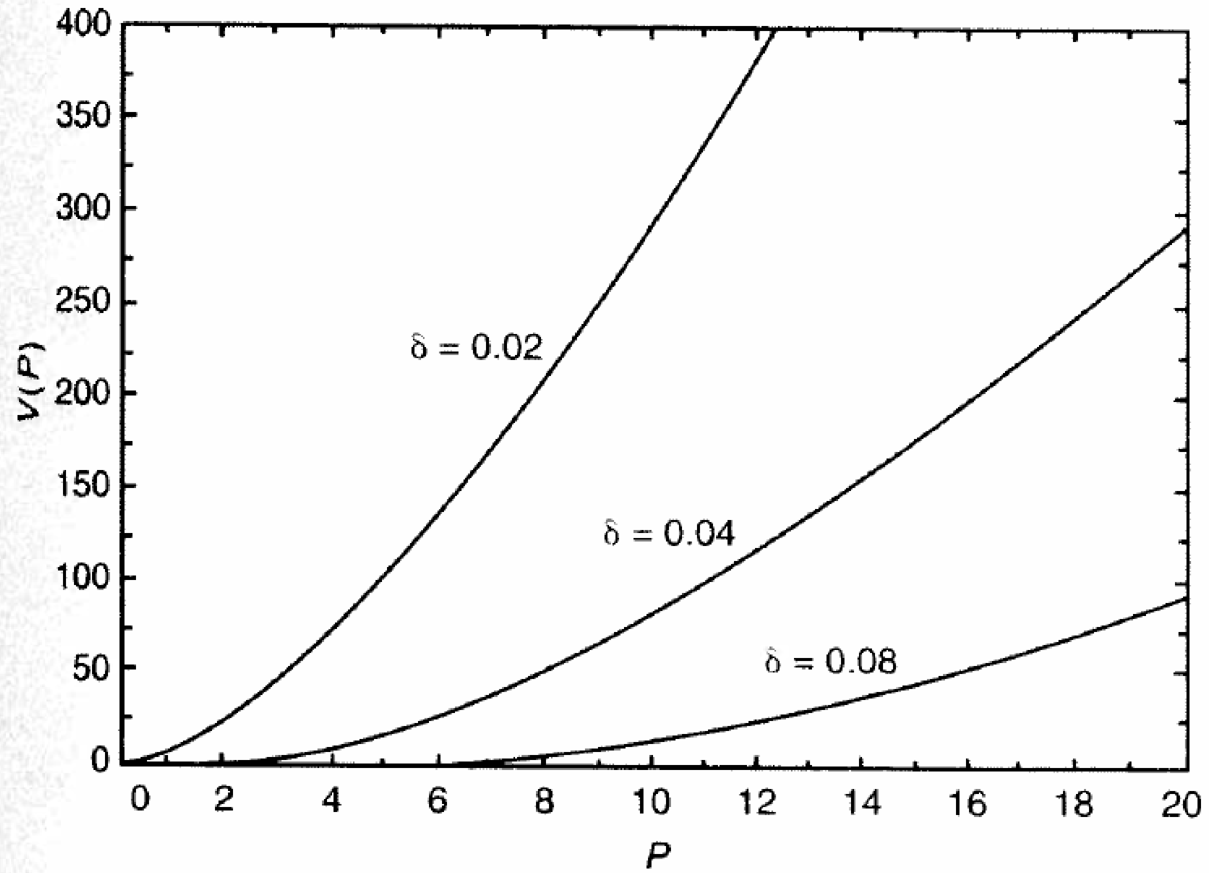


Figure 6.2. *Value of Project, $V(P)$, for $\delta = 0.02, 0.04, 0.08$*

(Note: $r = 0.04$, $\sigma = 0.2$, and $C = 10$)

Operating Costs and Temporary Suspension: Value of the Option to Invest

- Following the contingent claims approach, $F(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2}$
  - Boundary condition $F(0) = 0 \Rightarrow A_2 = 0$
- For $P < C$, it is never optimal to invest
  - Thus, VM and SP of $F(P)$ will occur for $P \geq C$, i.e., with $V(P) - I = B_2 P^{\beta_2} + \frac{P}{\delta} - \frac{C}{\rho} - I$
  - Use $A_1 (P^*)^{\beta_1} = B_2 (P^*)^{\beta_2} + \frac{P^*}{\delta} - \frac{C}{\rho} - I$ and $\beta_1 A_1 (P^*)^{\beta_1 - 1} = \beta_2 B_2 (P^*)^{\beta_2 - 1} + \frac{1}{\delta}$ to solve for P^* and A_1
  - Substitute to solve the following equation numerically: $(\beta_1 - \beta_2) B_2 (P^*)^{\beta_2} + (\beta_1 - 1)\frac{P^*}{\delta} - \beta_1 \left(\frac{C}{\rho} + I\right) = 0$
  - Solution for $\rho = 0.04$, $\delta = 0.04$, $\sigma = 0.20$, $I = 100$, and $C = 10$ (Figure 6.3)
  - $\beta_1 = 2$, $\beta_2 = -1$, $P^{*,nf} = 28$, $A_1^{nf} = 0.4464$, $P^* = 23.8$, and $A_1 = 0.4943$
  - Sensitivity analysis: $F(P)$ and P^* increase with σ (Figure 6.4)
  - But $F(P)$ decreases and P^* increases with δ (Figures 6.5 and 6.6)

Operating Costs and Temporary Suspension: Figure 6.3

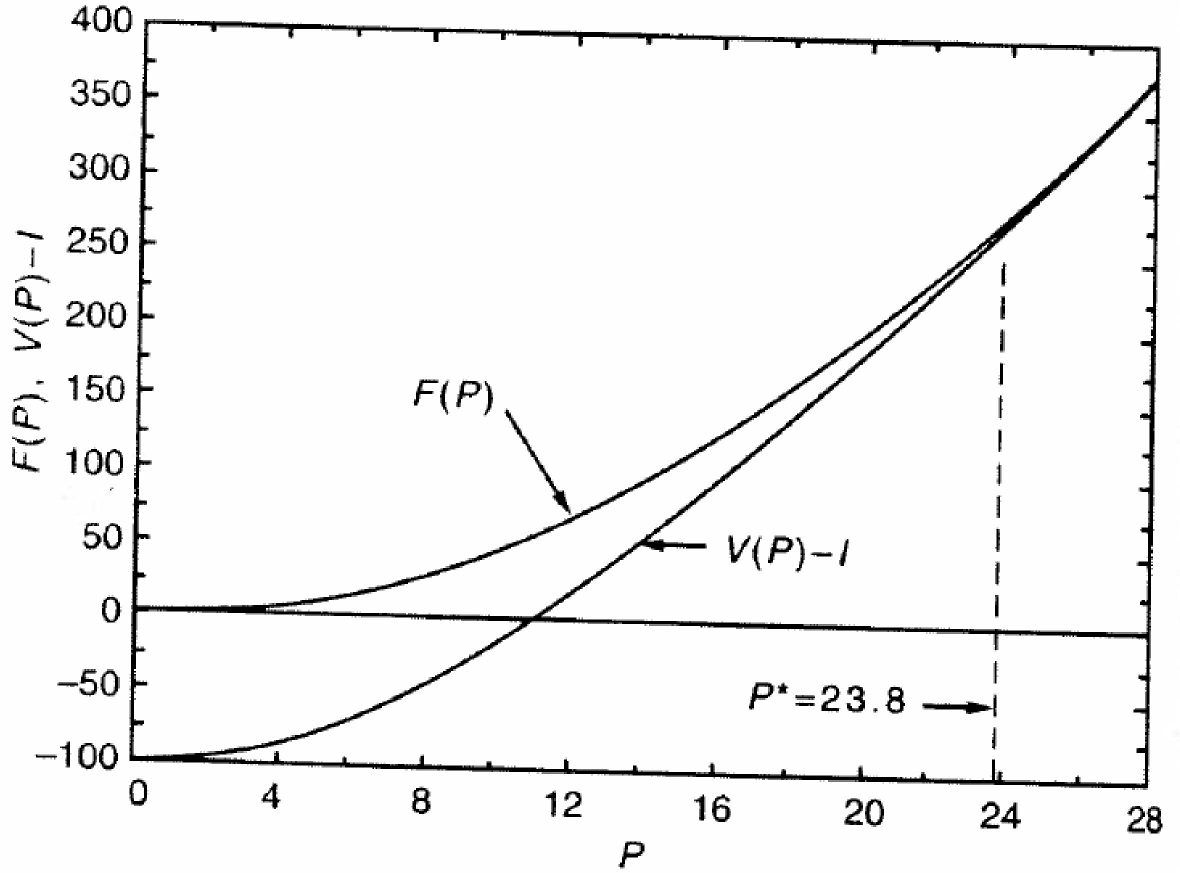


Figure 6.3. *Value of Investment Opportunity, $F(P)$, and $V(P) - I$*

(Note: $r = \delta = 0.04$, $\sigma = 0.2$, and $I = 100$)

Operating Costs and Temporary Suspension: Figure 6.4

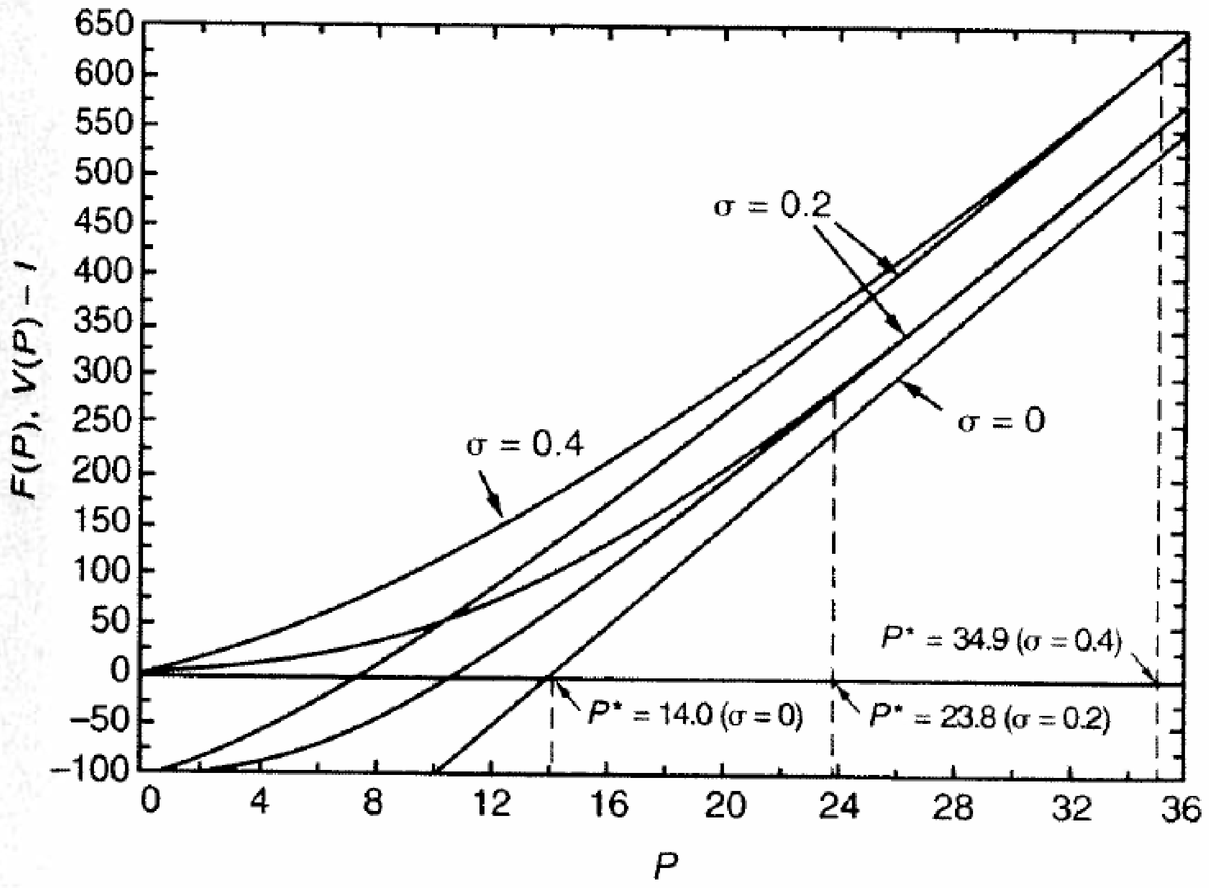


Figure 6.4. *Value of Investment Opportunity, $F(P)$, and $V(P) - I$, for $\sigma = 0$, 0.2, and 0.4*

Operating Costs and Temporary Suspension: Figure 6.5

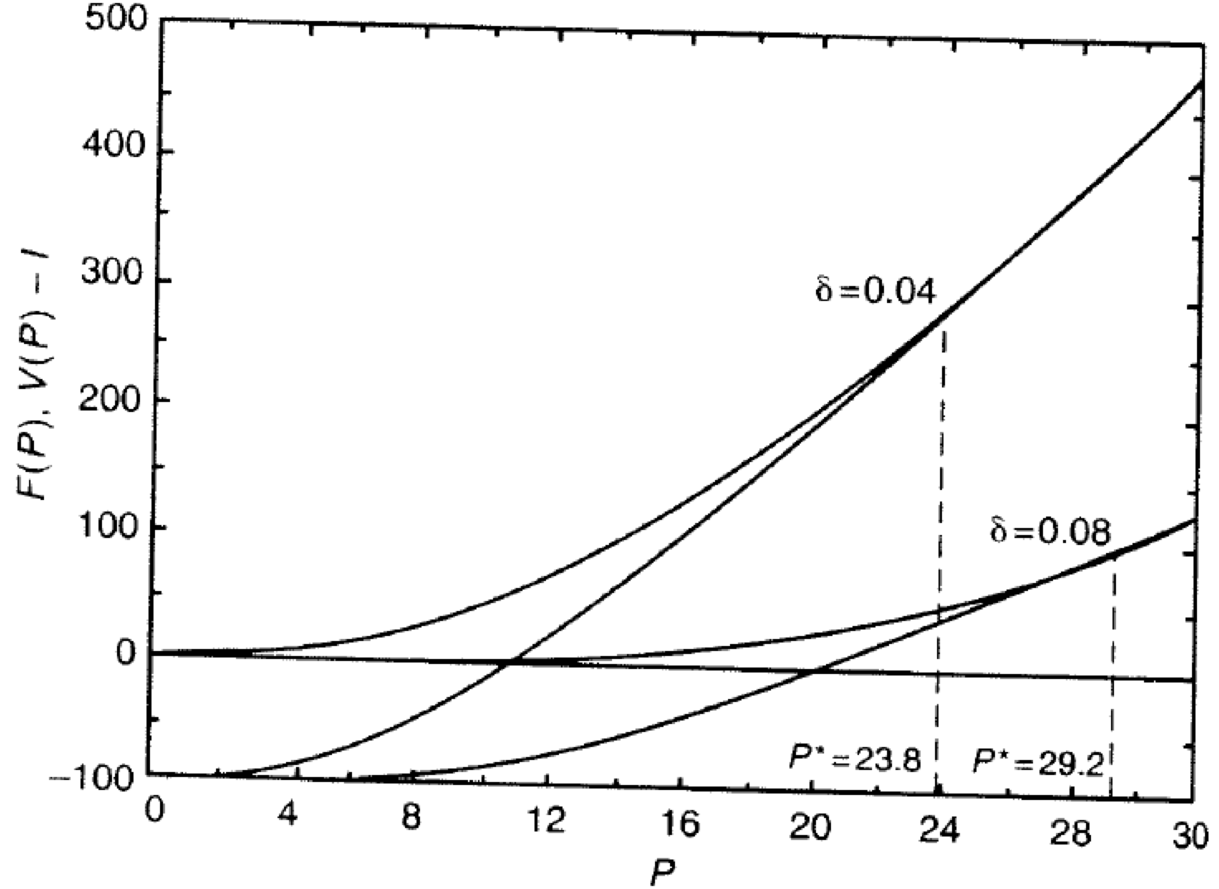
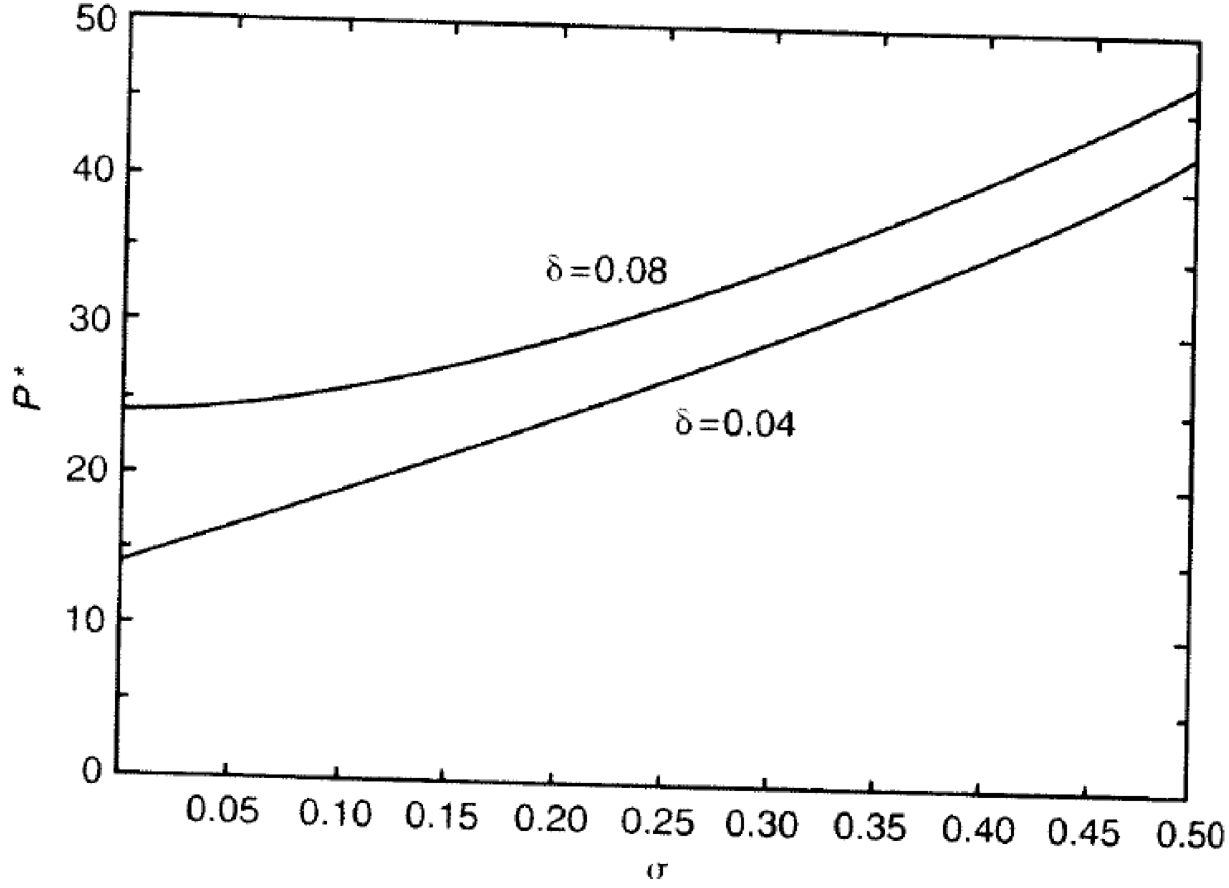


Figure 6.5. *Value of Investment Opportunity, F(P), and V(P)−I, for δ = 0.04 and 0.08*

Operating Costs and Temporary Suspension: Figure 6.6



Optimal Stopping Time Approach: Now-or-Never NPV

- Example from McDonald (2005): oil extraction under certainty at a rate of one barrel per year forever
  - Current price of oil is $P_0 = 15$, discount rate is $\rho = 0.05$, growth rate of oil is $\alpha = 0.01$, operating cost is $C = 8$, and investment cost is $I = 180$
- Is it optimal to extract the oil now?
  - Assuming that the price of oil grows exponentially, the NPV from immediate extraction is $V(P_0) = \int_0^\infty e^{-\rho t} \left\{ P_0 e^{\alpha t} - C \right\} dt - I = \frac{P_0}{\rho - \alpha} - \frac{C}{\rho} - I = 215 - 180 = 35$
  - Since $V(P_0) > 0$, it is optimal to extract
- But, would it not be better to wait longer?
- Investment cost is being discounted, and the value of the oil is growing

Optimal Stopping Time Approach: Deterministic NPV

★ Think instead about value of perpetual investment opportunity

- $F(P_0) = \max_T \int_T^\infty e^{-\rho t}\left\{P_0 e^{\alpha t} - C - \rho I\right\}dt = \max_T \frac{P_0}{\rho-\alpha}e^{(\alpha-\rho)T} - \frac{C}{\rho}e^{-\rho T} - Ie^{-\rho T}$
- $\Rightarrow T^* = \frac{1}{\alpha}\ln\left(\frac{C+\rho I}{P_0}\right) = 12.5163$
- Or, invest when $P_{T^*} = 17$
- Indeed, the initial value of the investment opportunity is $F(P_0) = 45.46 > 35 = V(P_0)$

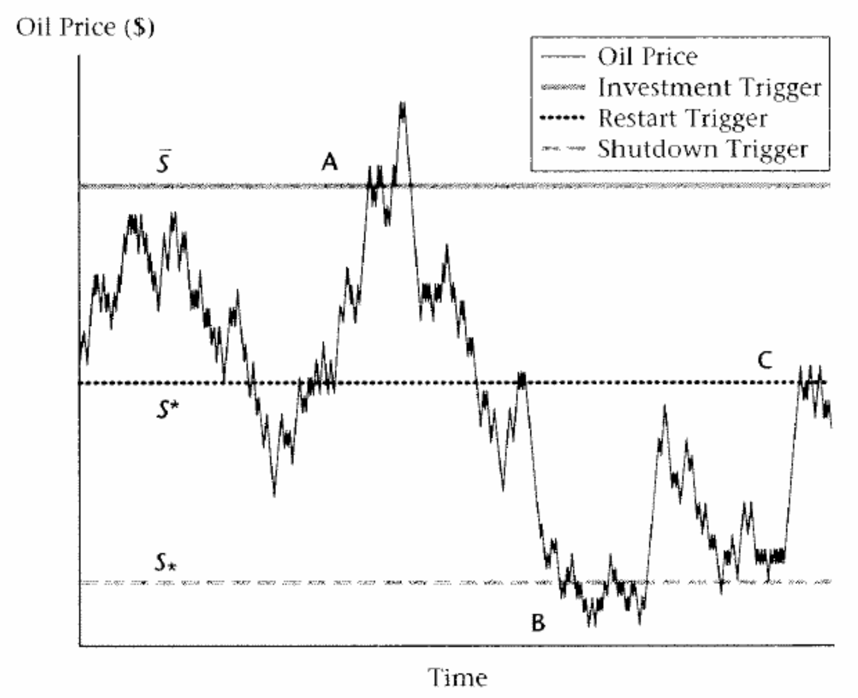
★ By delaying investment to the optimal time period, it is possible to maximise NPV

★ How does this work when the price is stochastic?

Optimal Investment under Uncertainty

★ Price process evolves according to a GBM, i.e., $dP_t = \alpha P_t dt + \sigma P_t dz_t$ with initial price $P_0 = p$

- Note that $(dP_t)^2 = \sigma^2 (P_t)^2 dt$



Optimal Investment under Uncertainty

- If the project were started now, then its expected NPV is $V(p) = \mathbb{E}_p \left[\int_0^\infty e^{-\rho t} \left\{ P_t - (C + \rho I) \right\} dt\right] = \frac{p}{\rho - \alpha} - \frac{C}{\rho} - I$

- Canonical real options problem:

$$F(p) = \sup_{\tau \in \mathcal{S}} \mathbb{E}_p \left[\int_\tau^\infty e^{-\rho t} \left\{ P_t - (C + \rho I) \right\} dt\right]$$

$$\Rightarrow F(p) = \sup_{\tau \in \mathcal{S}} \mathbb{E}_p \left[e^{-\rho \tau} V(P_\tau)\right] = \max_{P^* \geq p} \left\{ \left(\frac{p}{P^*}\right)^{\beta_1} V(P^*) \right\}$$

  - β_1 (β_2) is the positive (negative) root of $\frac{1}{2}\sigma^2 \zeta(\zeta - 1) + \alpha\zeta - \rho = 0$

Optimal Investment Threshold under Uncertainty

- Solve for optimal investment threshold, P^*:

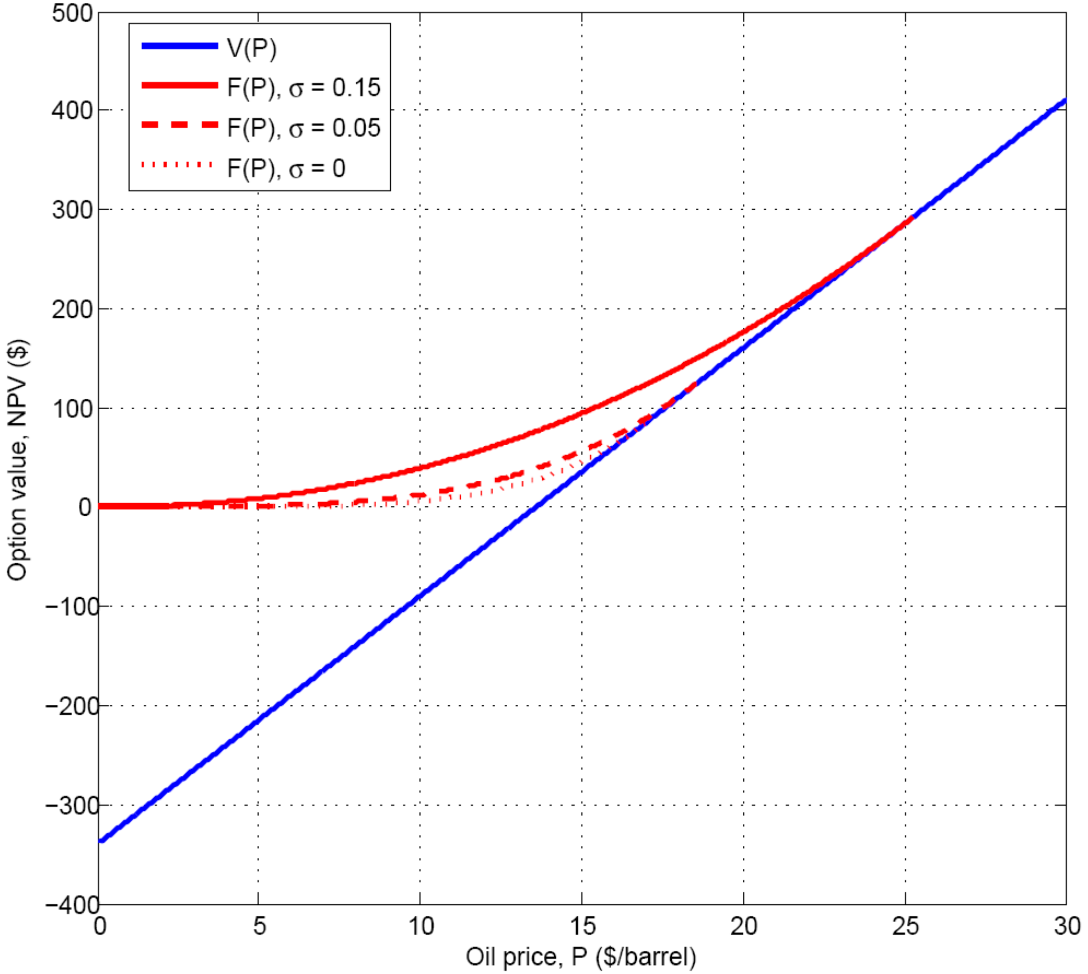
$$F(p) = \max_{P^* \geq p} \left\{ \left(\frac{p}{P^*} \right)^{\beta_1} V(P^*) \right\}$$

  - First-order necessary condition yields $P^* = \frac{\beta_1}{\beta_1 - 1}(\rho - \alpha)\left(\frac{C}{\rho} + I\right)$
  - Note that in the case without uncertainty, $\beta_1 = \frac{\rho}{\alpha} \Rightarrow P^* = C + \rho I$

- For a level of volatility of $\sigma = 0.15$, $P^* = 25.28$, and the value of the investment opportunity is $F(p) = 94.35$

- Compared to the case with certainty, the investment opportunity is worth more, but is also less likely to be exercised

Investment Thresholds and Values



Investment under Uncertainty with Abandonment

- If the project is abandoned after investment, then the expected incremental payoff is:

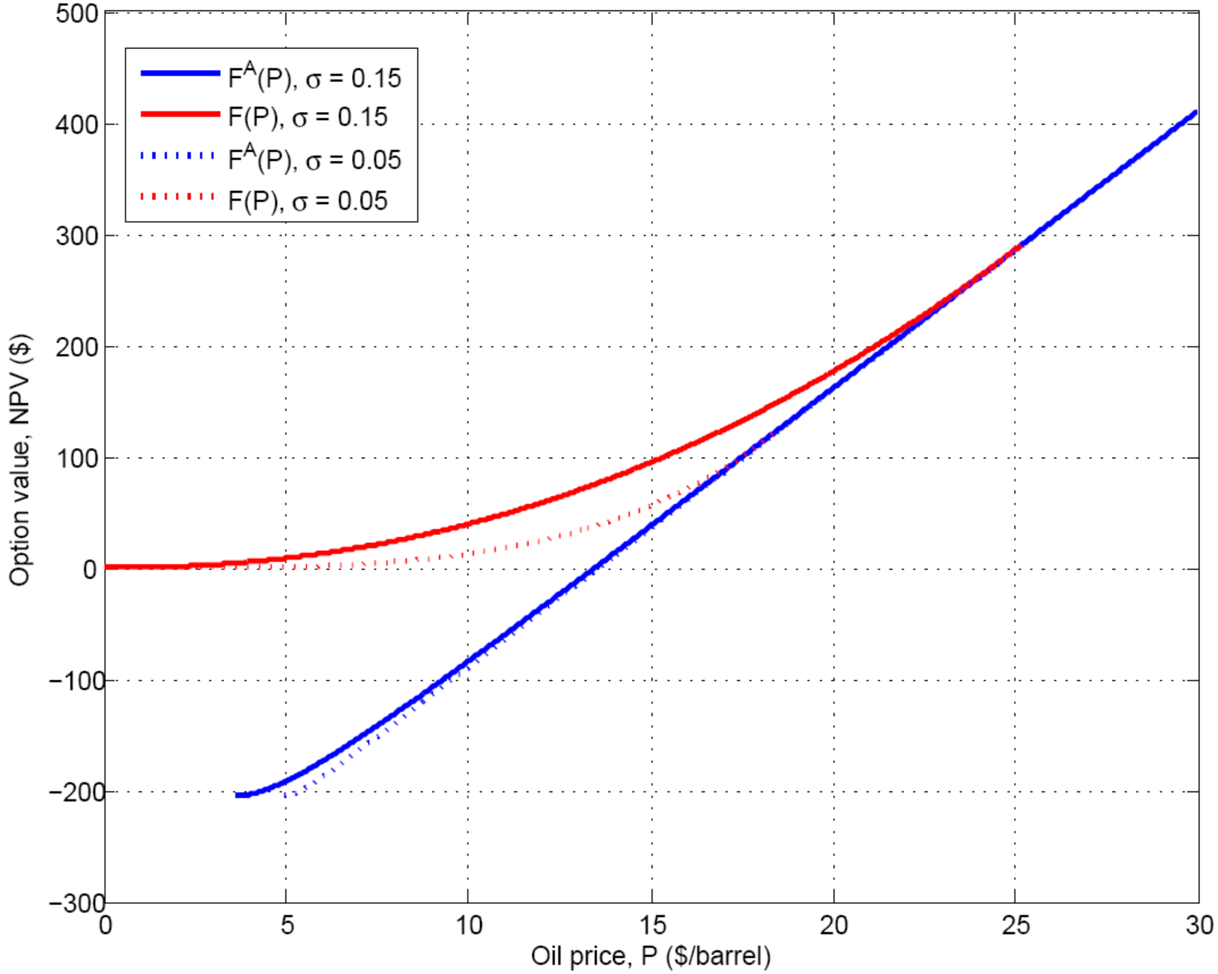
$$V^A(p) = \mathbb{E}_p\left[\int_0^\infty e^{-\rho t}\left\{(C-\rho K_s) - P_t\right\}dt\right] = \frac{C}{\rho} - K_s - \frac{p}{\rho-\alpha}$$

- Solve for optimal abandonment threshold, P_*:

$$F^A(p) = \max_{P_* \leq p}\left\{\left(\frac{p}{P_*}\right)^{\beta_2} V^A(P_*)\right\} + V(p)$$

  - First-order necessary condition yields $P_* = \frac{\beta_2}{\beta_2-1}(\rho-\alpha)\left(\frac{C}{\rho} - K_s\right)$
  - Solve numerically for P^*: $F(p) = \max_{P^* \geq p}\left\{\left(\frac{p}{P^*}\right)^{\beta_1}\left\{V(P^*) + \left(\frac{P^*}{P_*}\right)^{\beta_2} V^A(P_*)\right\}\right\}$

Investment Thresholds and Values with Abandonment



Investment under Uncertainty with Suspension and Resumption

- If the project is resumed from a suspended state, then the expected incremental payoff is:

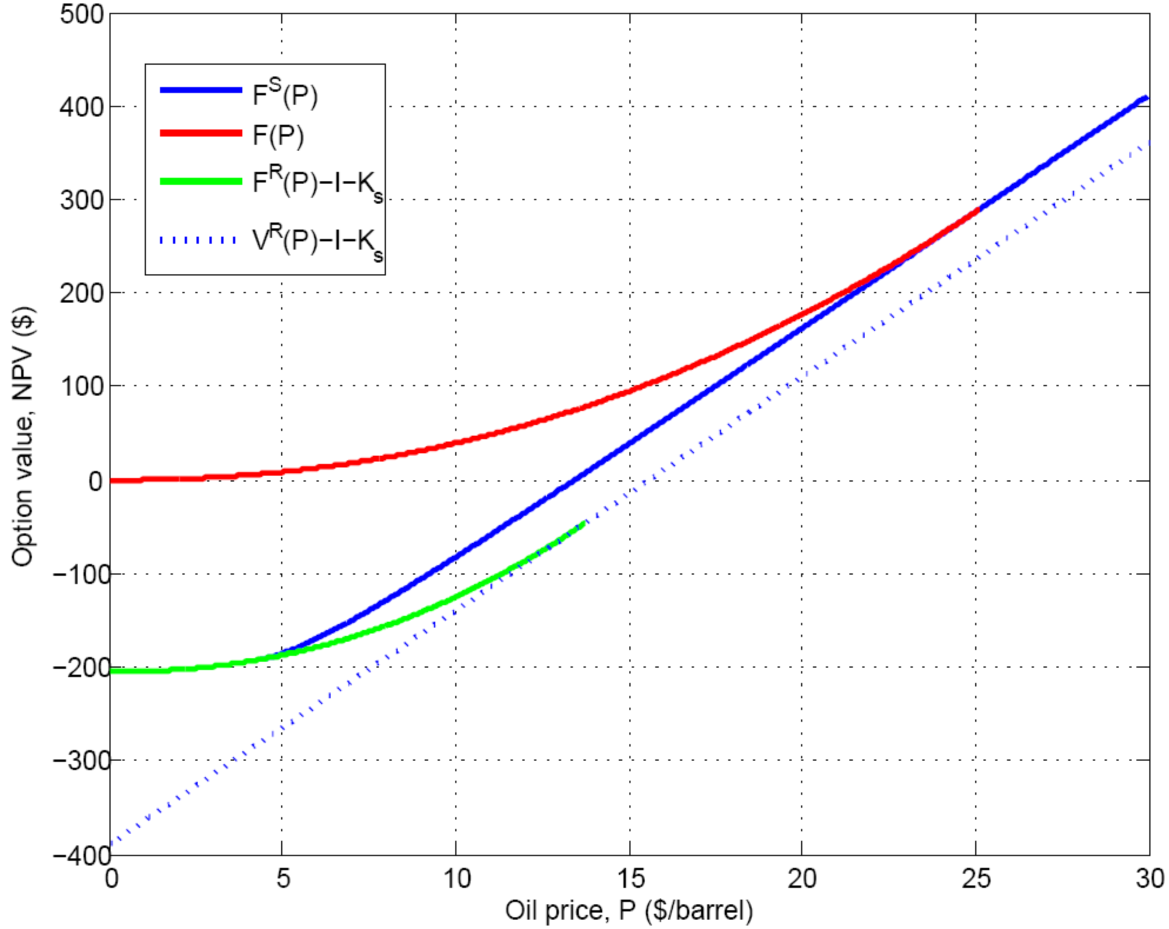
$$V^R(p) = \mathbb{E}_p\left[\int_0^\infty e^{-\rho t}\left\{P_t - (C + \rho K_r)\right\} dt\right] = \frac{p}{\rho - \alpha} - \frac{C}{\rho} - K_r$$

- Solve for optimal resumption threshold, P^{**}:

$$F^R(p) = \max_{P^{**} \geq p}\left\{\left(\frac{p}{P^{**}}\right)^{\beta_1} V^R(P^{**})\right\}$$

  - First-order necessary condition yields $P^{**} = \frac{\beta_1}{\beta_1 - 1}(\rho - \alpha)\left(\frac{C}{\rho} + K_r\right)$
  - Substitute P^{**} back into $F^S(p)$ to solve numerically for P_* and then repeat for $F(p)$ to obtain P^*

Investment Thresholds and Values with Resumption



Investment with Infinite Suspension and Resumption Options

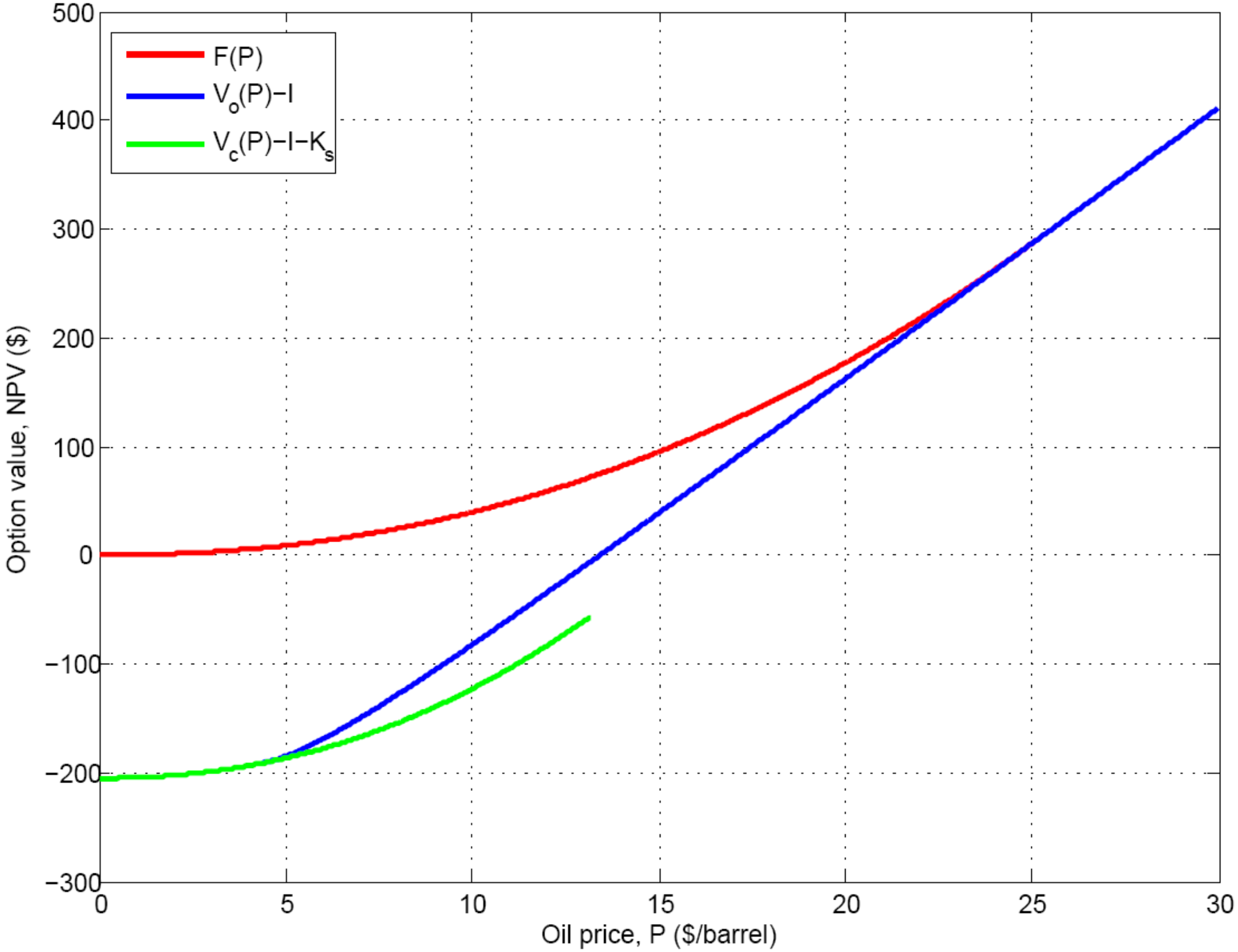
- Start with the expected value of a suspended project: $V_c(p, \infty, \infty; P_*, P^{**}) = \left(\frac{p}{P^{**}}\right)^{\beta_1} (V_o(P^{**}, \infty, \infty; P_*, P^{**}) - K_r)$

- Also note the expected value of an active project: $V_o(p, \infty, \infty; P_*, P^{**}) = \frac{p}{\rho-\alpha} - \frac{C}{\rho} + \left(\frac{p}{P_*}\right)^{\beta_2} \left(\frac{C}{\rho} - K_s - \frac{P_*}{\rho-\alpha} + V_c(P_*, \infty, \infty; P_*, P^{**})\right)$

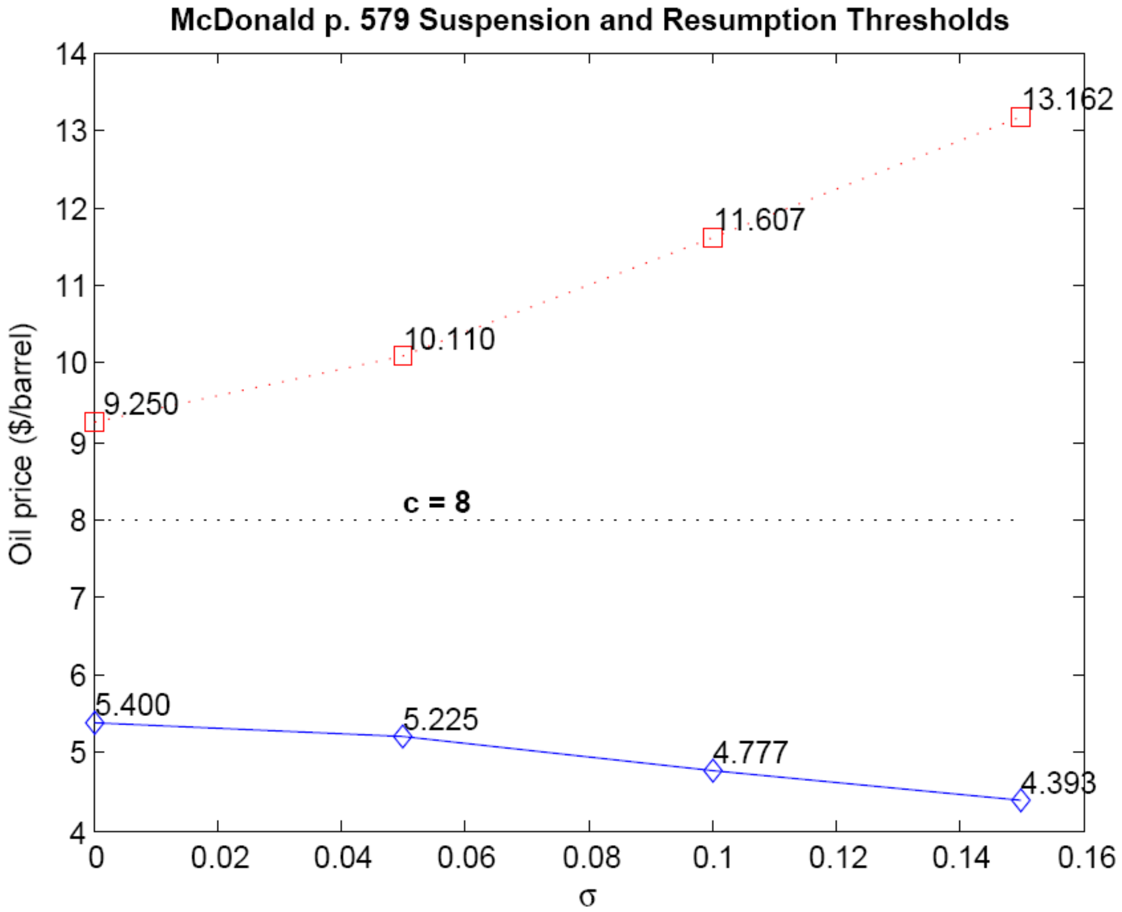
  - Solve the two equations numerically, i.e., start with initial thresholds and successively iterate until convergence

- Finally, solve for P^* numerically: $F(p, \infty, \infty; P_*, P^{**}) = \max_{P^* \geq p} \left(\frac{p}{P^*}\right)^{\beta_1} \{V_o(P^*, \infty, \infty; P_*, P^{**}) - I\}$

Investment Thresholds and Values with Complete Flexibility



Thresholds with Complete Flexibility



Numerical Results: Data from McDonald (2005)

- $P_0 = 15, C = 8, \rho = 0.05, \alpha = 0.01, I = 180, K_s = 25, K_r = 25$

σ	N_s	N_r	P_I	P_*	P^*	$F(P_0)$
0.05	0	0	18.5846	-	-	56.0527
0.10	0	0	21.5927	-	-	74.6799
0.15	0	0	25.2791	-	-	94.3469
0.05	1	0	18.5846	4.9396	-	56.0527
0.10	1	0	21.5821	4.2514	-	74.7062
0.15	1	0	25.1587	3.6315	-	94.6154
0.05	1	1	18.5846	5.2246	10.1122	56.0527
0.10	1	1	21.5784	4.7702	11.7489	74.7153
0.15	1	1	25.1233	4.3625	13.7548	94.6946
0.05	∞	∞	18.5846	5.2246	10.1104	56.0527
0.10	∞	∞	21.5784	4.7766	11.6070	74.7154
0.15	∞	∞	25.1219	4.3926	13.1619	94.6977

Seminar Outline

- ★ Mathematical Background (Dixit and Pindyck, 1994: chs. 3–4)
- ★ Investment and Operational Timing (Dixit and Pindyck, 1994: chs. 5–6 and McDonald, 2005: ch. 17)
- ★ Strategic Interactions (Huisman and Kort, 1999)
- ★ Capacity Switching (Siddiqui and Takashima, 2011)

Topic Outline

- Classification of setups
- Pre-emptive setting
- Non-pre-emptive setting

Interaction of Game Theory and Real Options

- Fudenberg and Tirole (1985) treat a duopoly with pre-emption over timing in a deterministic model

- Huisman and Kort (1999) extend this to reflect market uncertainty to find that the incentive to delay in real options may be reduced due to competition

- Possible settings: cooperative and non-cooperative (pre-emptive and non-pre-emptive)

Duopoly Assumptions

- Each decision-maker has the perpetual right to start a project at any time for deterministic investment cost, I
- Price process evolves according to a GBM, i.e., $dP_t = \alpha P_t dt + \sigma P_t dz_t$ with initial price $P_0 > 0$
  - Subjective interest rate is ρ
  - An active project produces one unit of output per year forever
- $R_t = P_t D(Q_t)$ is the project's revenue given $Q_t = 0, 1, 2$ active firms in the industry and $D(1) > D(2)$
- $\tau_i^j \equiv \min\left\{t \geq 0 : P_t \geq P_{\tau_i^j}\right\}$, $j = L, F$ and $i = m, p, n$

Formulation 1: Monopoly

- Value function if monopolist has invested $(P_0 \geq P_{\tau_m^j})$: $V_m^j(P_0) = \mathbb{E}_{P_0}\left[\int_0^\infty e^{-\rho t}\left\{P_t D(1) - \rho I)\right\} dt\right]$
  - $V_m^j(P_0) = \frac{P_0 D(1)}{\rho - \alpha} - I$
- Value function if monopolist is waiting to invest, i.e., $P_0 < P_{\tau_m^j}$: $V_m^j(P_0) = \sup_{\tau_m^j \in \mathcal{S}} \mathbb{E}_{P_0}\left[\int_{\tau_m^j}^\infty e^{-\rho t}\left\{P_t D(1) - \rho I\right\} dt\right]$
  - $V_m^j(P_0) = \sup_{\tau_m^j \in \mathcal{S}} \mathbb{E}_{P_0}\left[e^{-\rho \tau_m^j}\right]\left(\frac{P_0 D(1)}{\rho - \alpha} - I\right)$
- Monopolist's entry threshold: $P_{\tau_m^j} = \left(\frac{\beta_1}{\beta_1 - 1}\right)\frac{\rho I}{D(1)}$

Formulation 2: Pre-Emptive Duopoly

- Follower's problem:
  - If $P_0 \geq P_{\tau_p^F}$: $V_p^F(P_0) = \frac{P_0 D(2)}{\rho - \alpha} - I$
  - Else: $V_p^F(P_0) = \sup_{\tau_p^F \in \mathcal{S}} \mathbb{E}_{P_0}\left[e^{-\rho \tau_p^F}\right]\left(\frac{P_{\tau_p^F} D(2)}{\rho - \alpha} - I\right)$
  - Entry threshold: $P_{\tau_p^F} = \left(\frac{\beta_1}{\beta_1 - 1}\right)\frac{\rho I}{D(2)}$
- Leader's problem:
  - Value function for $P_0 \geq P_{\tau_p^F}$ is the same as the follower's
  - Else: $V_p^L(P_0) = \frac{P_0 D(1)}{\rho - \alpha} - I + \left(\frac{P_0}{P_{\tau_p^F}}\right)^{\beta_1}\left[\frac{P_{\tau_p^F}(D(2) - D(1))}{\rho - \alpha}\right]$
  - Find τ_p^L by setting $V_p^L(P_{\tau_p^L}) = V_p^F(P_{\tau_p^L})$

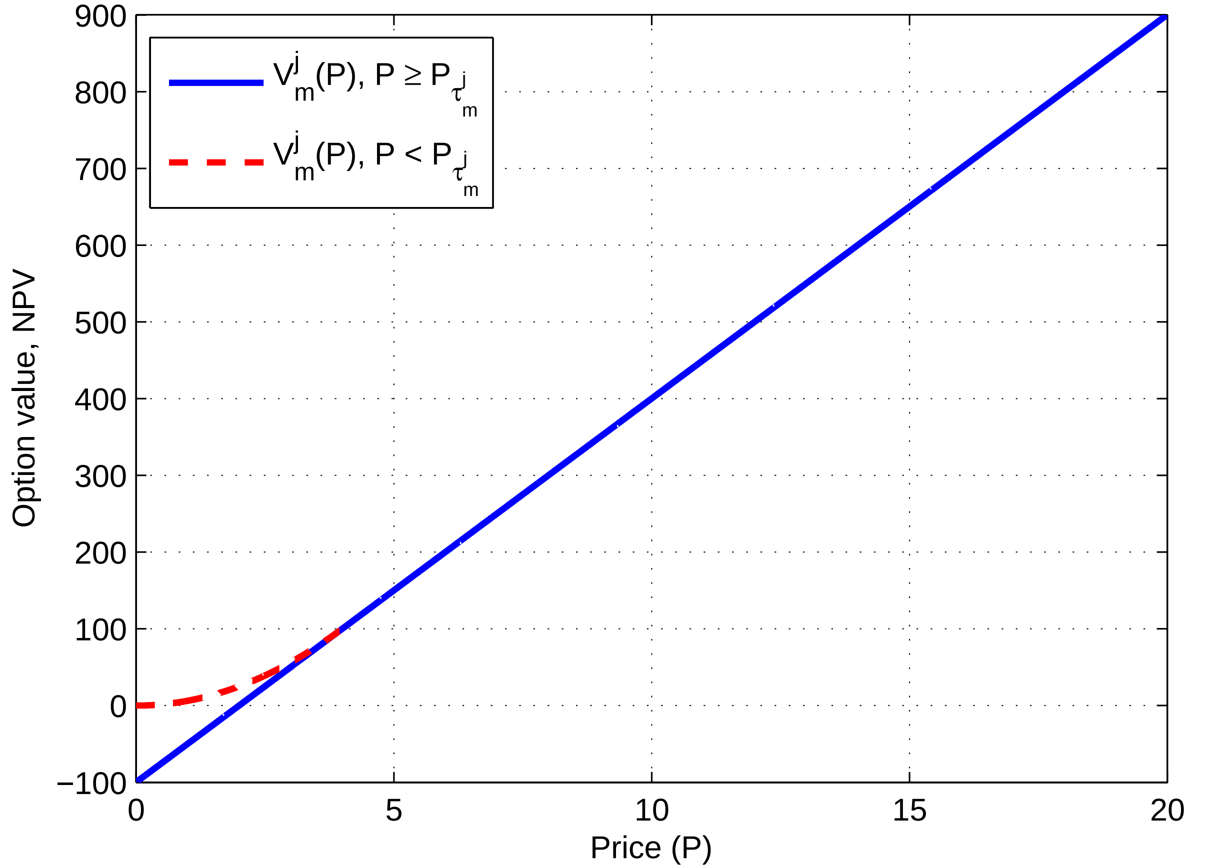
Formulation 3: Non-Pre-Emptive Duopoly

- Follower's problem is the same as under the pre-emptive duopoly framework, i.e., $V_n^F(P_0) = V_p^F(P_0)$ and $P_{\tau_p^F} = P_{\tau_n^F}$

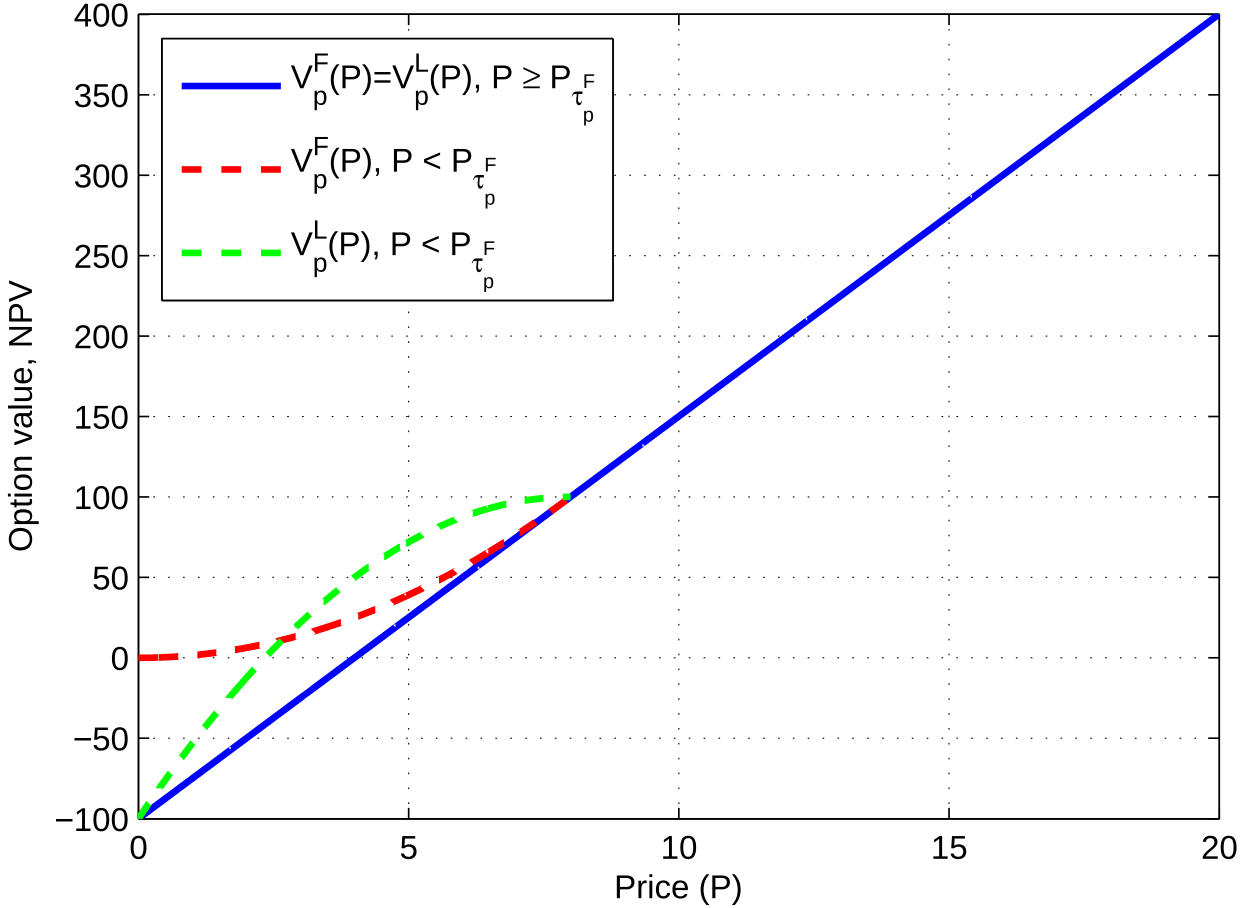
- Leader's problem:
  - Leader's value function for $P_0 \geq P_{\tau_n^F}$ is the same as in the pre-emptive case, i.e., $V_n^L(P_0) = V_p^L(P_0)$
  - Leader's value function for $P_{\tau_n^L} \leq P_0 < P_{\tau_n^F}$ is also the same as in the pre-emptive case
  - Else: $V_n^L(P_0) = \max_{P_{\tau_n^L} \geq P_0} \left(\frac{P_0}{P_{\tau_n^L}}\right)^{\beta_1} \left[\frac{P_{\tau_n^L} D(1)}{\rho-\alpha} - I \right.$ $\left. + \left(\frac{P_{\tau_n^L}}{P_{\tau_p^F}}\right)^{\beta_1} \left[\frac{P_{\tau_p^F}(D(2)-D(1))}{\rho-\alpha}\right]\right]$
  - Optimal entry threshold for the leader in the non-pre-emptive case is the same as that for a monopolist: $P_{\tau_n^L} = \left(\frac{\beta_1}{\beta_1-1}\right)\frac{\rho I}{D(1)}$

Numerical Example: Monopoly

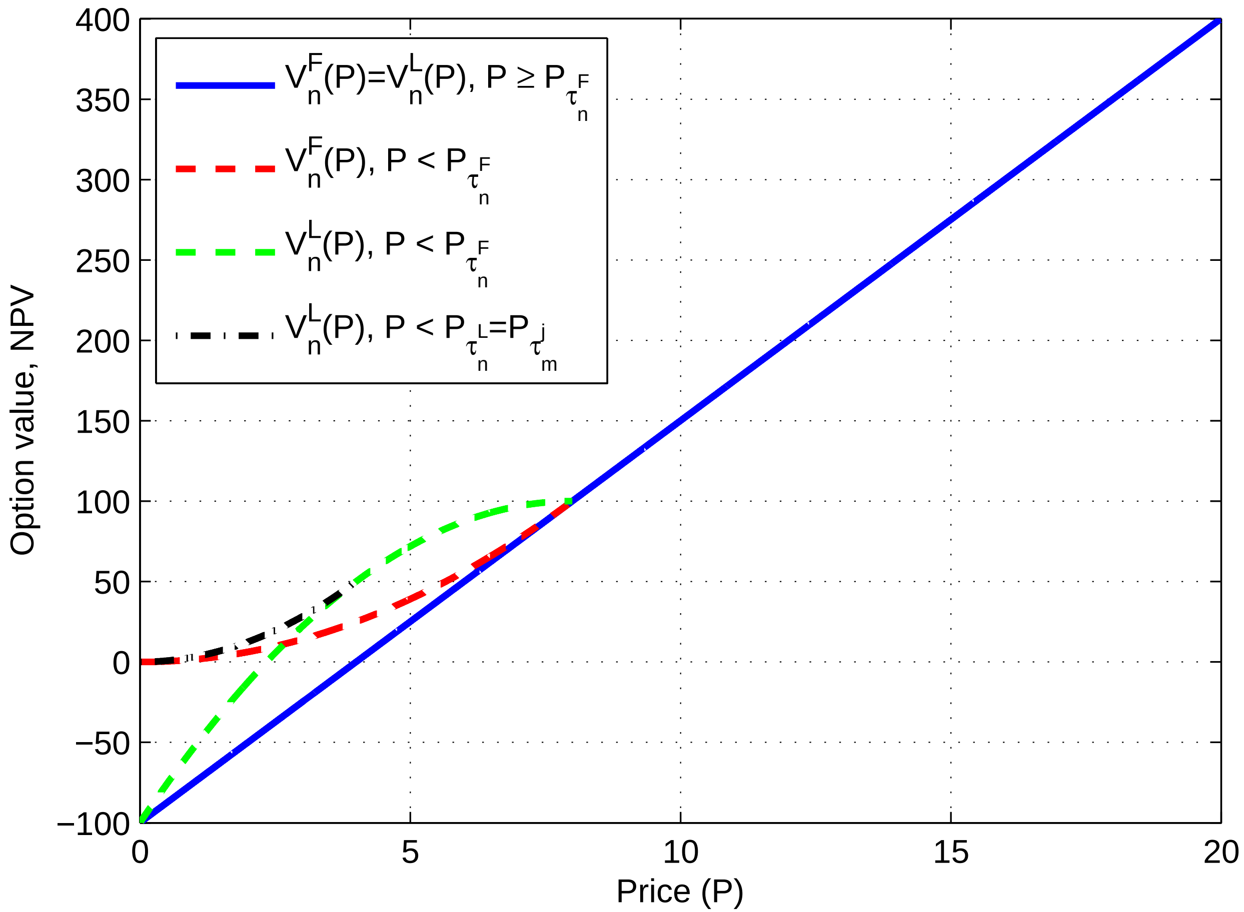
$\sigma = 0.20,\ \rho = 0.04,\ \alpha = 0,\ I = 100,\ D(1) = 2,\ D(2) = 1$



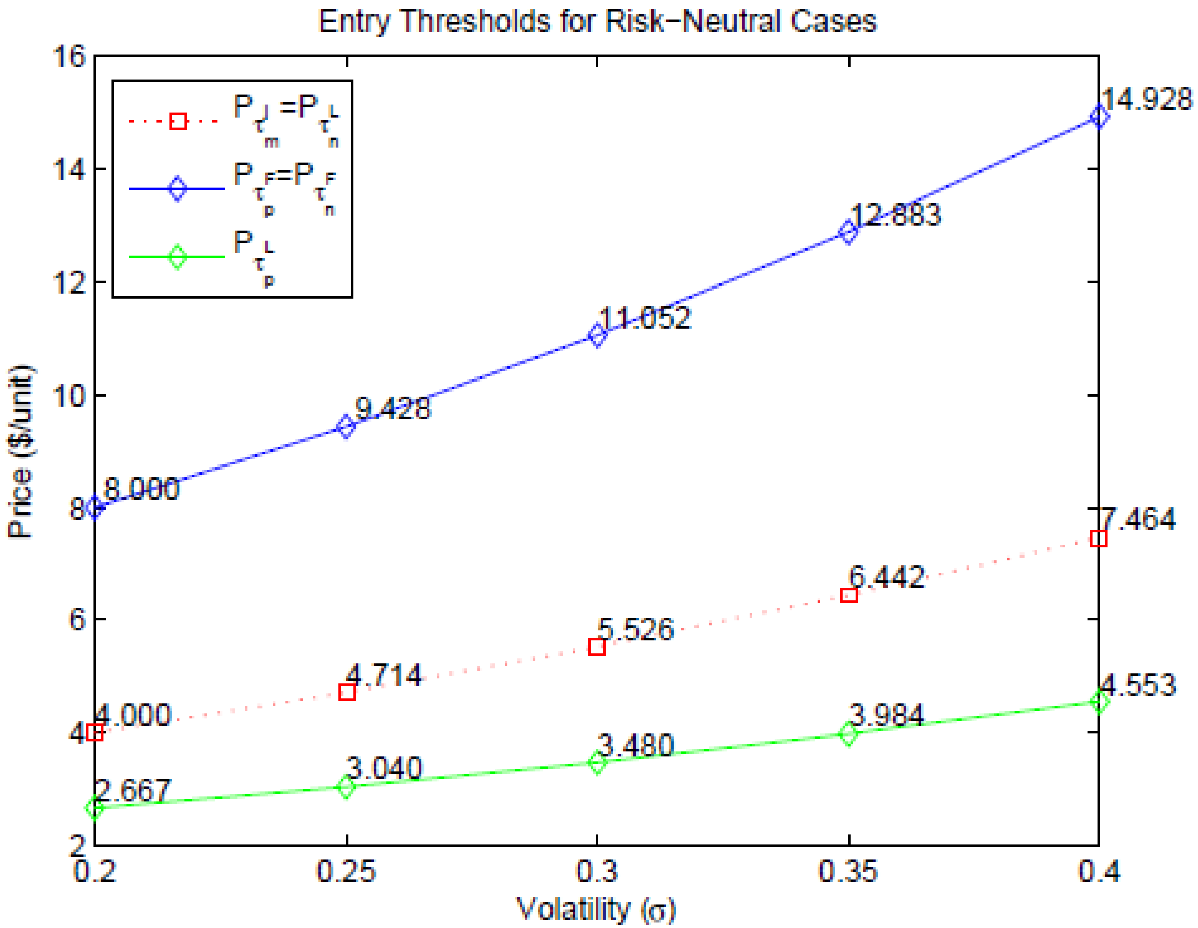
Numerical Example: Pre-Emptive Duopoly



Numerical Example: Non-Pre-Emptive Duopoly

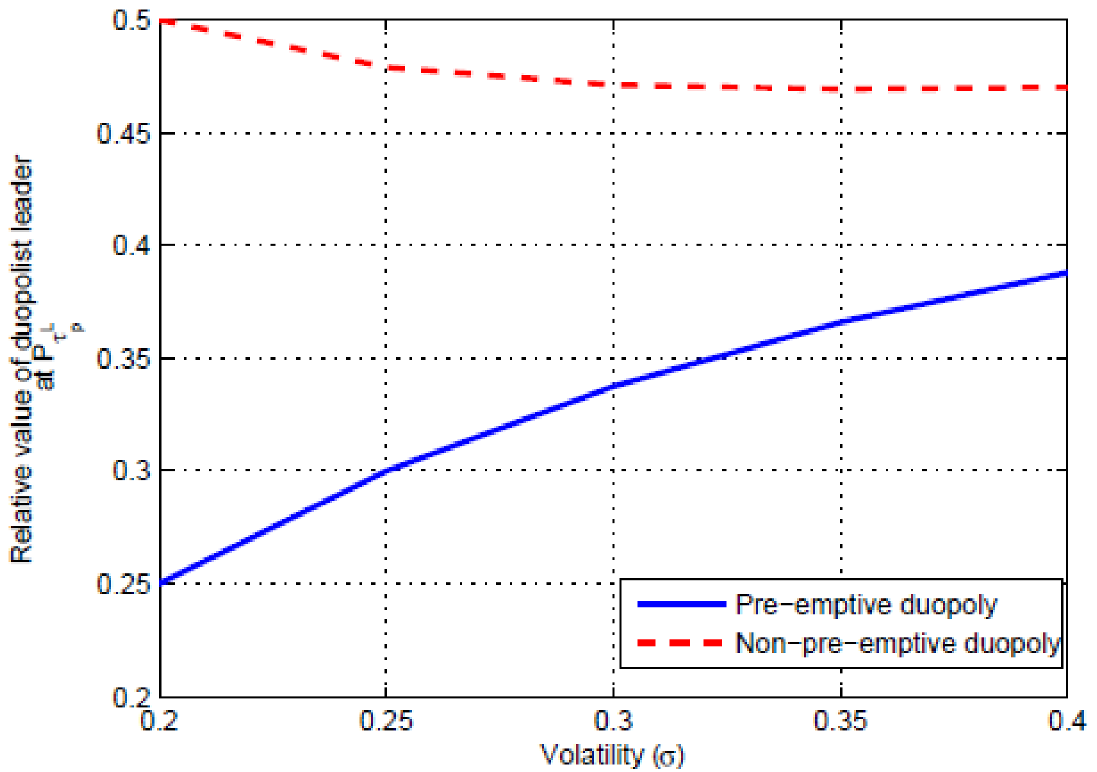


Numerical Example: Entry Threshold Sensitivity Analysis



Numerical Example: Option Value Sensitivity Analysis

$$\frac{V_p^L(P_{\tau_p^L})}{V_m^j(P_{\tau_p^L})} \text{ or } \frac{V_n^L(P_{\tau_p^L})}{V_m^j(P_{\tau_p^L})}$$



Seminar Outline

★ Mathematical Background (Dixit and Pindyck, 1994: chs. 3–4)

★ Investment and Operational Timing (Dixit and Pindyck, 1994: chs. 5–6 and McDonald, 2005: ch. 17)

★ Strategic Interactions (Huisman and Kort, 1999)

★ Capacity Switching (Siddiqui and Takashima, 2011)

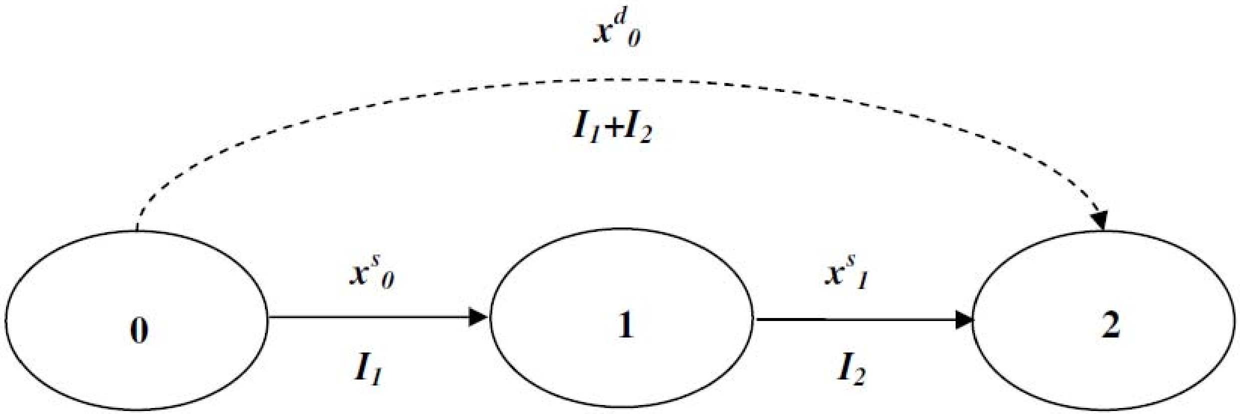
Topic Outline

★ Monopoly

★ Spillover duopoly

★ Proprietary duopoly

Monopoly Setup



- Direct strategy: obtain project of size K_2 for an investment cost of $I_1 + I_2$
- Sequential strategy: invest in size K_1 before deciding to switch to a project with a higher capacity, K_2 (total cost is still $I_1 + I_2$)
- Market shock: $dx_t = \alpha x_t dt + \sigma x_t dz_t$, where $\alpha \geq 0$ and $\sigma \geq 0$
- $P_t = x_t D(\kappa_t)$ (in \$/unit), where κ_t is the installed capacity (in units/annum) at time t and $D(\kappa_t)$ is the demand parameter given the installed capacity at time t (strictly decreasing)
- $\rho > \alpha$

Monopoly: Direct Strategy

- $V_2^d(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\rho t} K_2 x_t D_2 dt\right] - I_1 - I_2 = \frac{x K_2 D_2}{\rho - \alpha} - I_2 - I_1$
- Value function in state 0: $V_0^d(x) = A_0^d x^{\beta_1}$
- Value-matching and smooth-pasting conditions:
  - $V_0^d(x_0^d) = V_2^d(x_0^d)$
  - $\frac{dV_0^d}{dx}\big|_{x=x_0^d} = \frac{dV_2^d}{dx}\big|_{x=x_0^d}$
- Solution yields $x_0^d = \left(\frac{\beta_1}{\beta_1 - 1}\right)\frac{(I_1+I_2)(\rho-\alpha)}{K_2 D_2}$ and $A_0^d = \frac{x_0^{d^{-\beta_1}}(I_1+I_2)}{\beta_1 - 1}$

Monopoly: Sequential Strategy

★ $V_1^s(x) = \frac{xK_1D_1}{\rho-\alpha} - I_1 + A_1^s x^{\beta_1}$ if $x < x_1^s$ and $V_1^s(x) = V_2^s(x)$ otherwise

★ State-1 value-matching and smooth-pasting conditions:

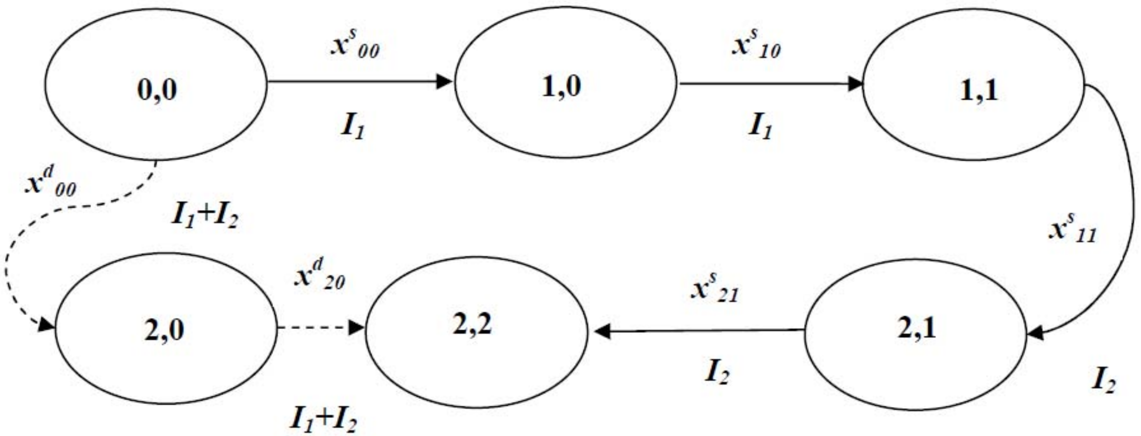
- $V_1^s(x_1^{s-}) = V_1^s(x_1^{s+})$
- $\frac{dV_1^s}{dx}\Big|_{x=x_1^{s-}} = \frac{dV_1^s}{dx}\Big|_{x=x_1^{s+}}$

★ Solution yields $x_1^s = \left(\frac{\beta_1}{\beta_1-1}\right)\frac{I_2(\rho-\alpha)}{[K_2D_2-K_1D_1]} > x_0^d$ and $A_1^s = \frac{x_1^{s\,-\beta_1}I_2}{\beta_1-1} < A_0^d$

★ Value function in state 0: $V_0^s(x) = A_0^s x^{\beta_1}$

- VM and SP conditions lead to $x_0^s = \left(\frac{\beta_1}{\beta_1-1}\right)\frac{I_1(\rho-\alpha)}{K_1D_1} < x_0^d$ and $A_0^s = A_1^s + \frac{x_0^{s\,-\beta_1}I_1}{\beta_1-1}$

Spillover Duopoly Setup



- Symmetric non-pre-emptive duopoly with spillover knowledge
- Direct strategy: obtain project of size K_2 for an investment cost of $I_1 + I_2$ before follower makes similar investment
- Sequential strategy: invest in size K_1 before waiting for follower's entry
- Additional assumptions: $0 < D_{22} < D_{21} < D_{20} < D_{11} < D_{10} = D_1$, $K_2 D_{22} > K_1 D_{21}$, $K_2 D_{21} > K_1 D_{11}$, and $\frac{1}{2}(K_1 + K_2) D_{21} > K_1 D_{11}$

Spillover Duopoly: Direct Strategy

★ Value functions: $V_{22}^{j,d}(x) = \frac{xK_2D_{22}}{\rho-\alpha} - I_2 - I_1$, $V_{20}^{L,d}(x) = \frac{xK_2D_{20}}{\rho-\alpha} - I_2 - I_1 + A_{20}^{L,d}x^{\beta_1}$, $V_{20}^{F,d}(x) = A_{20}^{F,d}x^{\beta_1}$, and $V_{00}^{j,d}(x) = A_{00}^{j,d}x^{\beta_1}$

★ VM and SP conditions:

- $V_{20}^{F,d}(x_{20}^d) = V_{22}^{F,d}(x_{20}^d)$
- $\frac{dV_{20}^{F,d}}{dx}\big|_{x=x_{20}^d} = \frac{dV_{22}^{F,d}}{dx}\big|_{x=x_{20}^d}$
- $V_{20}^{L,d}(x_{20}^d) = V_{22}^{L,d}(x_{20}^d)$
- $V_{00}^{j,d}(x_{00}^d) = \frac{1}{2}\left[V_{20}^{L,d}(x_{00}^d) + V_{20}^{F,d}(x_{00}^d)\right]$
- $\frac{dV_{00}^{j,d}}{dx}\big|_{x=x_{00}^d} = \frac{1}{2}\left[\frac{dV_{20}^{L,d}}{dx}\big|_{x=x_{00}^d} + \frac{dV_{20}^{F,d}}{dx}\big|_{x=x_{00}^d}\right]$

Spillover Duopoly: Direct Strategy Solutions

- $x_{20}^{d} = \left(\frac{\beta_1}{\beta_1 - 1}\right) \frac{(I_1 + I_2)(\rho - \alpha)}{K_2 D_{22}}$
- $A_{20}^{F,d} = \frac{{x_{20}^{d}}^{-\beta_1}(I_1 + I_2)}{\beta_1 - 1}$
- $A_{20}^{L,d} = \frac{{x_{20}^{d}}^{-\beta_1}(I_1 + I_2)(D_{22} - D_{20})\beta_1}{(\beta_1 - 1)D_{22}}$
- $x_{00}^{d} = \left(\frac{\beta_1}{\beta_1 - 1}\right) \frac{(I_1 + I_2)(\rho - \alpha)}{K_2 D_{20}} = x_0^d$
- $A_{00}^{j,d} = \frac{1}{2}\left[A_{20}^{L,d} + A_{20}^{F,d} + \frac{{x_{00}^{d}}^{-\beta_1}(I_1 + I_2)}{\beta_1 - 1}\right]$

Spillover Duopoly: Sequential Strategy

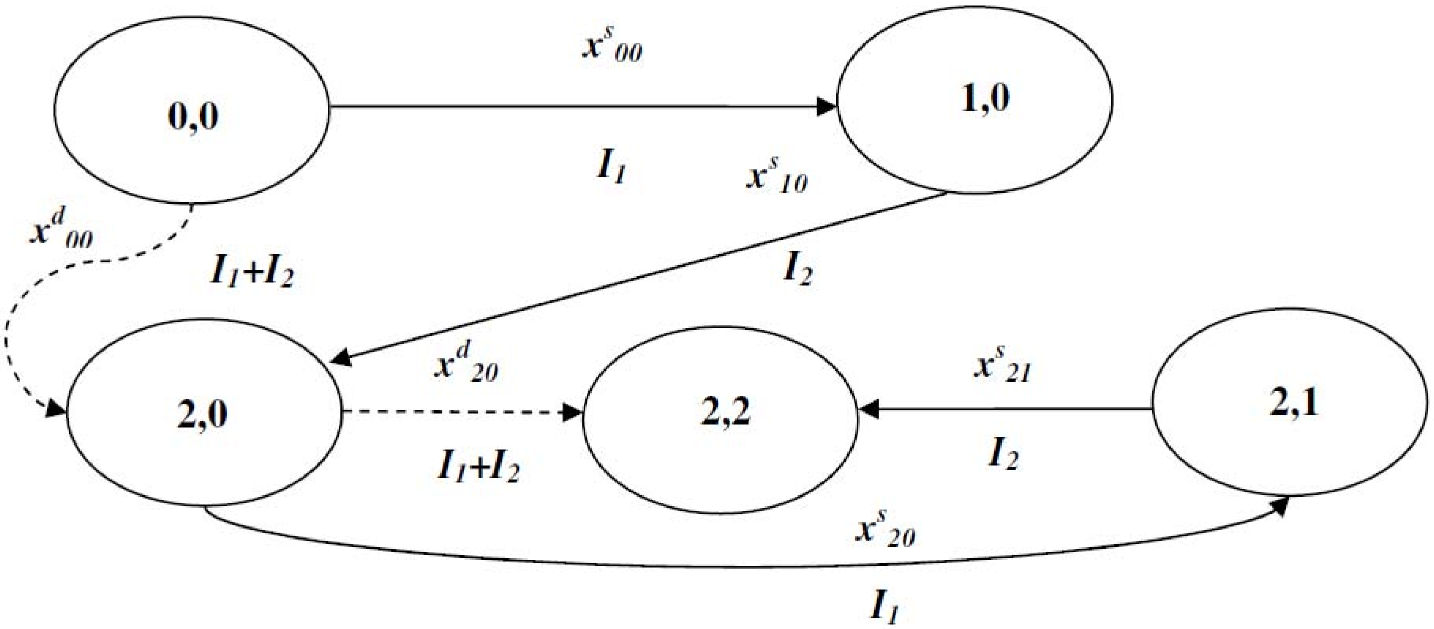
- Value functions: $V_{22}^{j,d}(x) = \frac{xK_2D_{22}}{\rho-\alpha} - I_2 - I_1$, $V_{21}^{L,s}(x) = \frac{xK_2D_{21}}{\rho-\alpha} - I_1 - I_2 + A_{21}^{L,s}x^{\beta_1}$, $V_{21}^{F,s}(x) = \frac{xK_1D_{21}}{\rho-\alpha} - I_1 + A_{21}^{F,s}x^{\beta_1}$, $V_{11}^{j,s}(x) = \frac{xK_1D_{11}}{\rho-\alpha} - I_1 + A_{11}^{j,s}x^{\beta_1}$, $V_{10}^{L,s}(x) = \frac{xK_1D_{10}}{\rho-\alpha} - I_1 + A_{10}^{L,s}x^{\beta_1}$, $V_{10}^{F,s}(x) = A_{10}^{F,s}x^{\beta_1}$, $V_{00}^{j,s}(x) = A_{00}^{j,s}x^{\beta_1}$

- Some VM and SP conditions:
  - $V_{21}^{F,s}(x_{21}^s) = V_{22}^{F,s}(x_{21}^s)$
  - $\frac{dV_{21}^{F,s}}{dx}|_{x=x_{21}^s} = \frac{dV_{22}^{F,s}}{dx}|_{x=x_{21}^s}$
  - $V_{21}^{L,d}(x_{21}^s) = V_{22}^{L,s}(x_{21}^s)$
  - $V_{11}^{j,s}(x_{11}^s) = \frac{1}{2}\left[V_{21}^{L,s}(x_{11}^s) + V_{21}^{F,s}(x_{11}^s)\right]$
  - $\frac{dV_{11}^{j,s}}{dx}|_{x=x_{11}^s} = \frac{1}{2}\left[\frac{dV_{21}^{L,s}}{dx}|_{x=x_{11}^s} + \frac{dV_{21}^{F,s}}{dx}|_{x=x_{11}^s}\right]$

Spillover Duopoly: Sequential Strategy Solutions

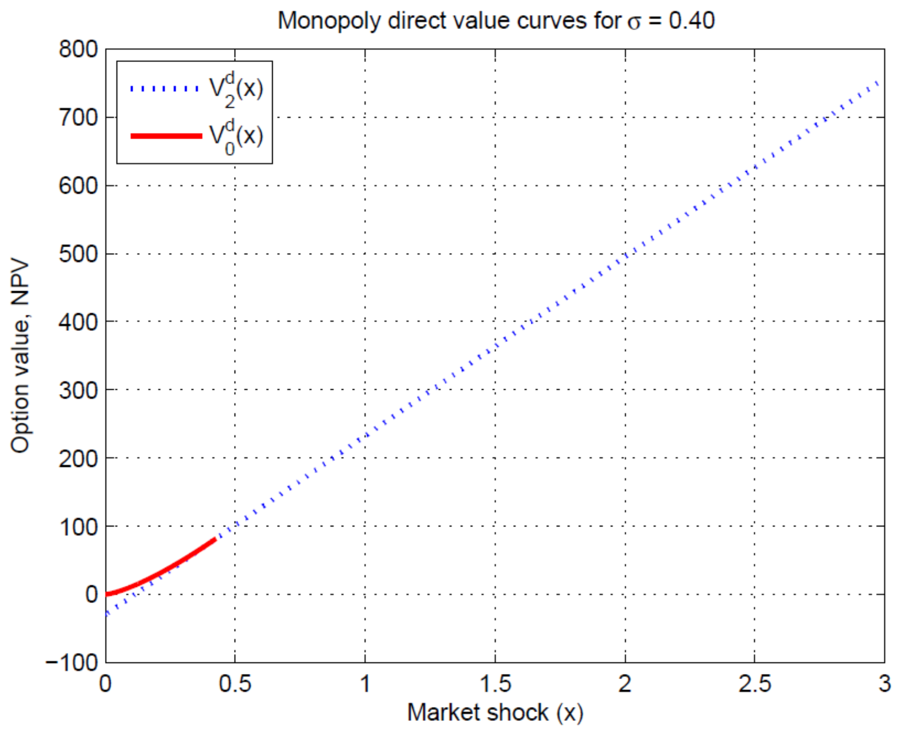
- $x_{21}^{s} = \left(\frac{\beta_1}{\beta_1 - 1}\right) \frac{I_2(\rho - \alpha)}{[K_2 D_{22} - K_1 D_{21}]}$
- $A_{21}^{F,s} = \frac{{x_{21}^{s}}^{-\beta_1} I_2}{\beta_1 - 1}$
- $A_{21}^{L,s} = \frac{{x_{21}^{s}}^{-\beta_1} I_2 \beta_1}{\beta_1 - 1} \left[\frac{K_2 D_{22} - K_2 D_{21}}{K_2 D_{22} - K_1 D_{21}}\right]$
- $x_{11}^{s} = \left(\frac{\beta_1}{\beta_1 - 1}\right) \frac{I_2(\rho - \alpha)}{[(K_1 + K_2) D_{21} - 2 K_1 D_{11}]}$
- $A_{11}^{j,s} = \frac{1}{2}\left(A_{21}^{L,s} + A_{21}^{F,s} + \frac{(x_{11}^{s})^{-\beta_1} I_2}{\beta_1 - 1}\right)$
- $x_{10}^{s} = \left(\frac{\beta_1}{\beta_1 - 1}\right) \frac{I_1(\rho - \alpha)}{K_1 D_{11}}$
- $A_{10}^{F,s} = A_{11}^{j,s} + \frac{{x_{10}^{s}}^{-\beta_1} I_1}{\beta_1 - 1}$
- $x_{00}^{s} = \left(\frac{\beta_1}{\beta_1 - 1}\right) \frac{I_1(\rho - \alpha)}{K_1 D_{10}} = x_0^{s}$
- $A_{00}^{j,s} = \frac{1}{2}\left(A_{10}^{L,s} + A_{10}^{F,s} + \frac{{x_{00}^{s}}^{-\beta_1} I_1}{\beta_1 - 1}\right)$

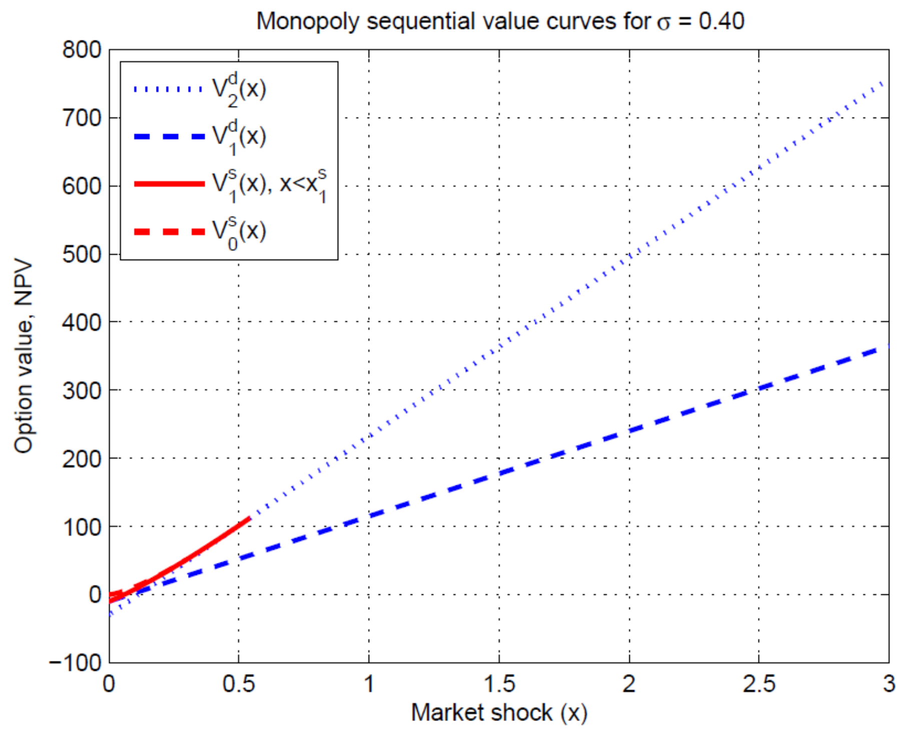
Proprietary Duopoly Setup



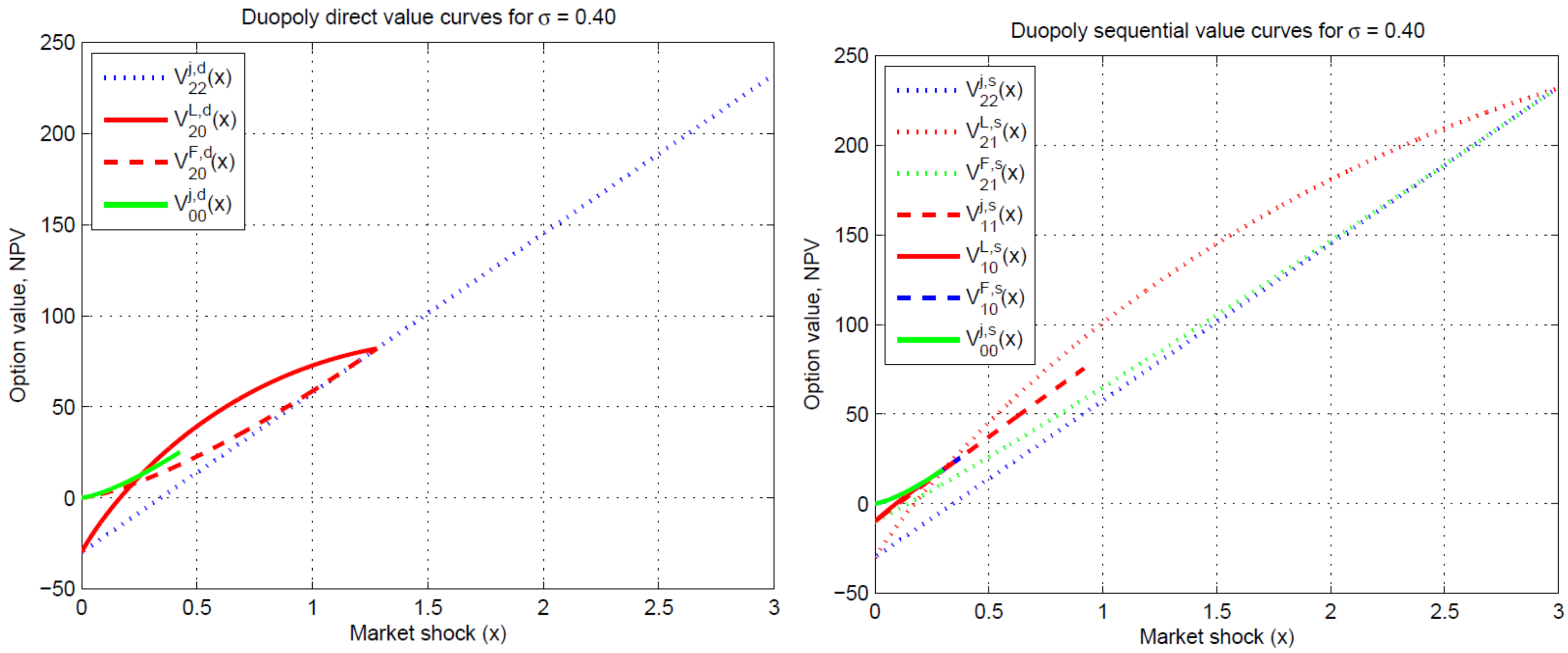
Numerical Example: Monopoly

$\sigma = 0.40,\ \rho = 0.04,\ \alpha = 0,\ I_1 = 10,\ I_2 = 20,\ K_1 = 1,\ K_2 = 3.5,\ D_{10} = 5,\ D_{11} = 4,\ D_{20} = 3,\ D_{21} = 2.5,\ D_{22} = 1$

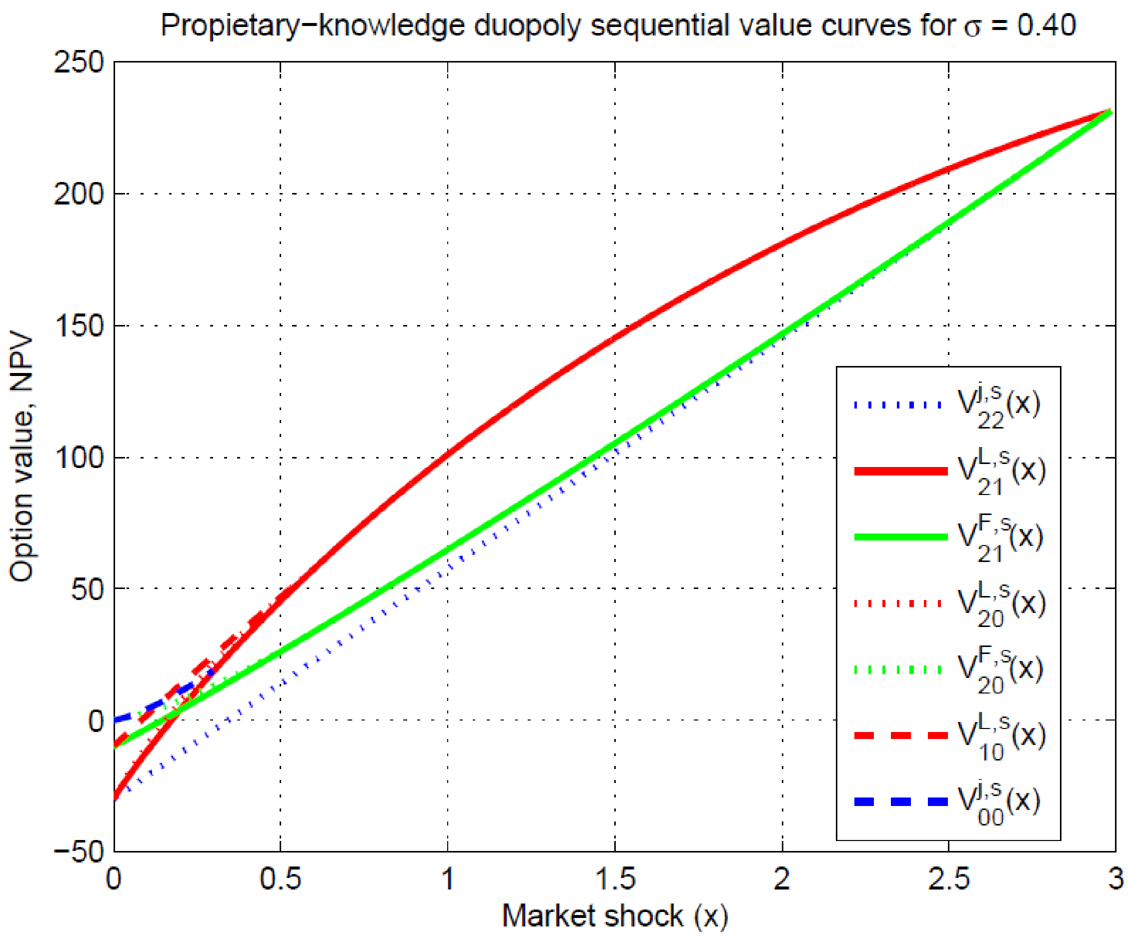




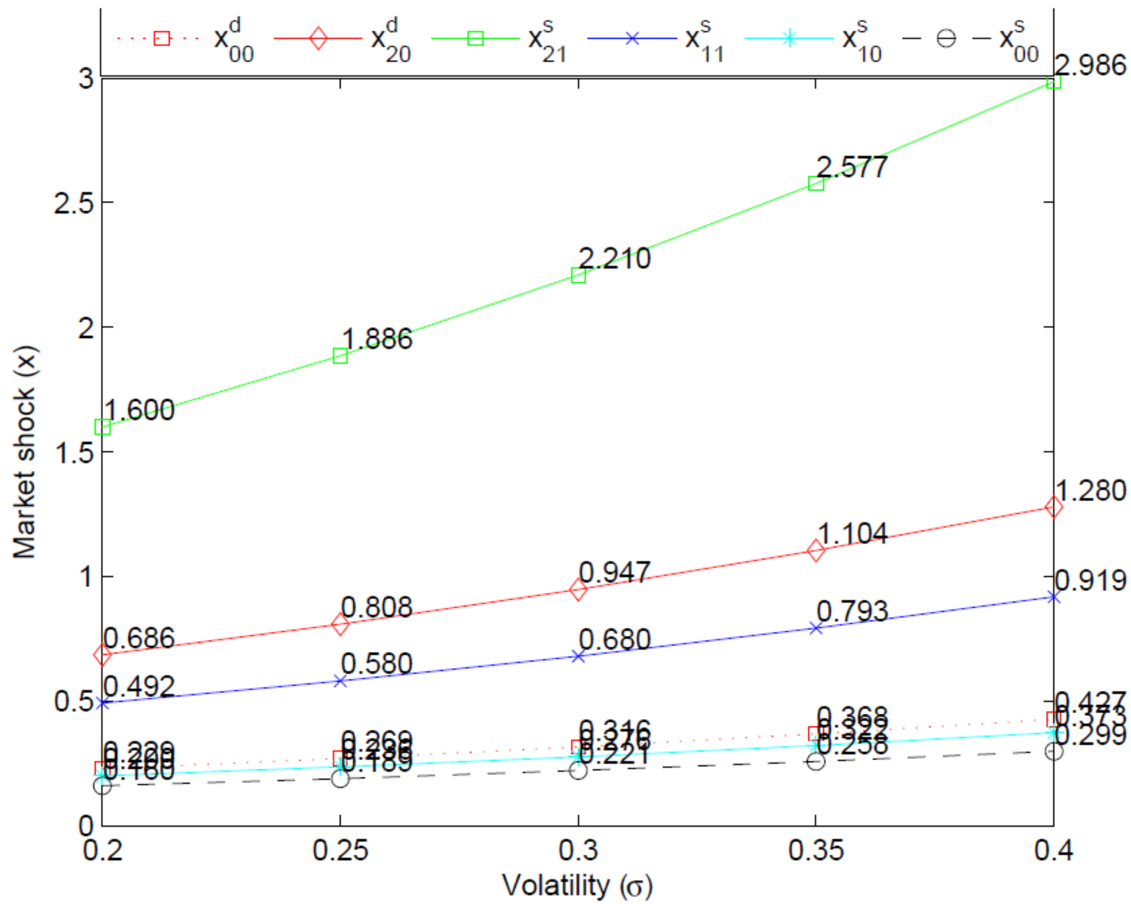
Numerical Example: Spillover Duopoly



Numerical Example: Proprietary Duopoly

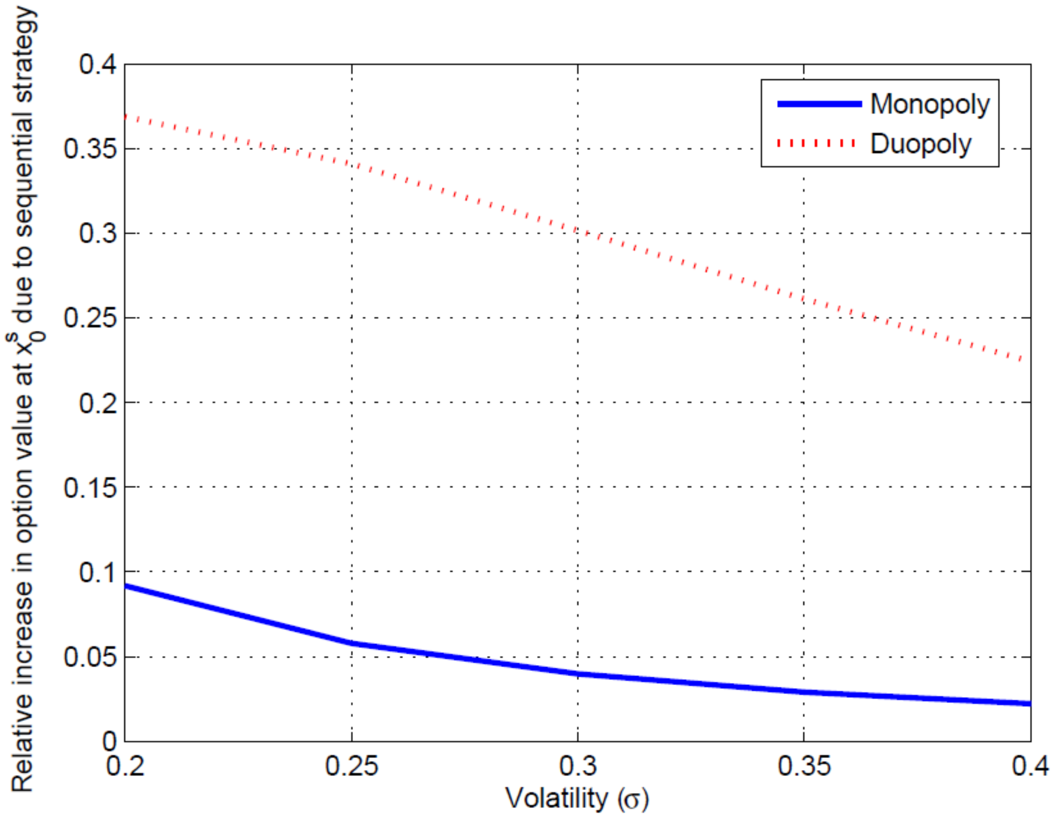


Numerical Example: Spillover Duopoly Thresholds



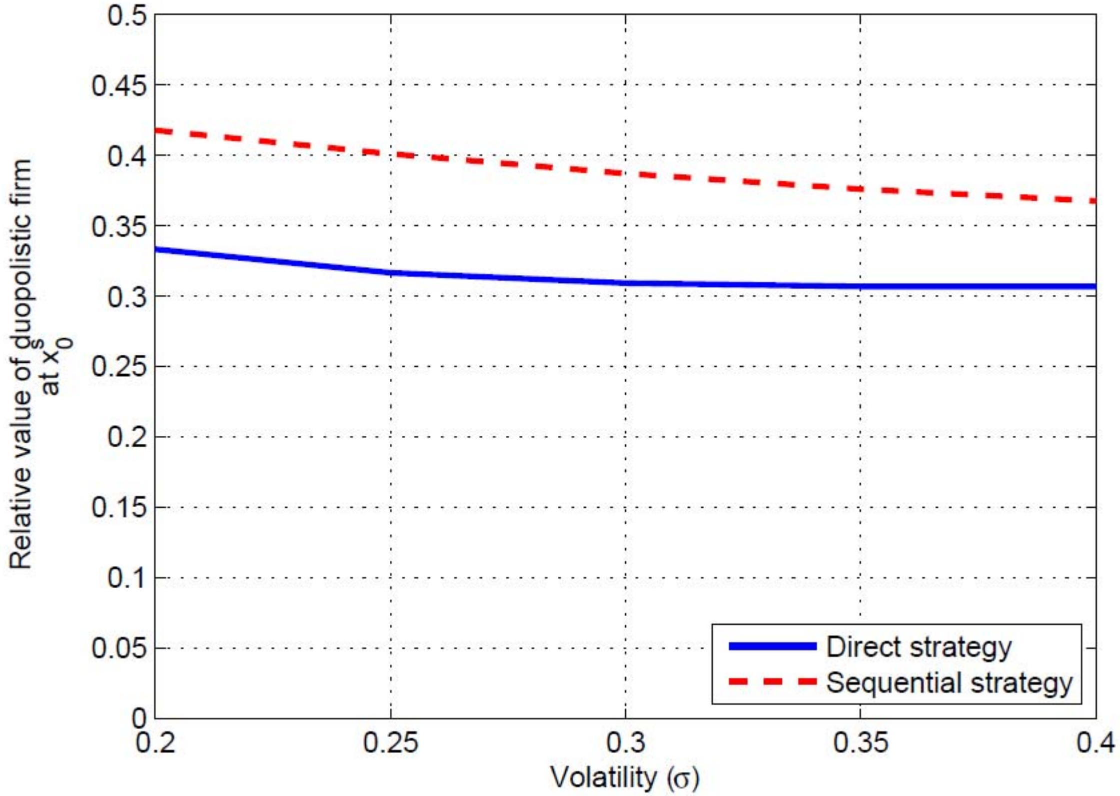
Numerical Example: Spillover Duopoly Value of Flexibility

$$\frac{V_0^s(x_0^s) - V_0^d(x_0^s)}{V_0^d(x_0^s)}$$



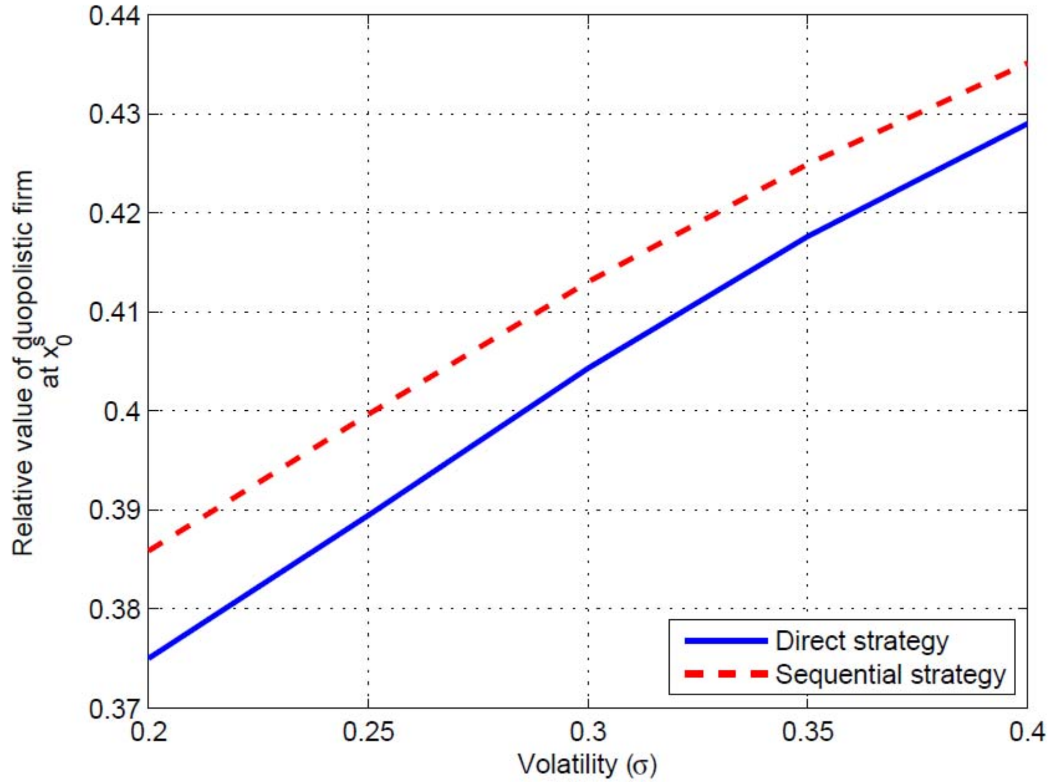
Numerical Example: Spillover Duopoly Effect of Competition

$$\frac{V_{00}^{j,d}(x_0^s)}{V_0^d(x_0^s)} \text{ or } \frac{V_{00}^{j,s}(x_0^s)}{V_0^s(x_0^s)}$$



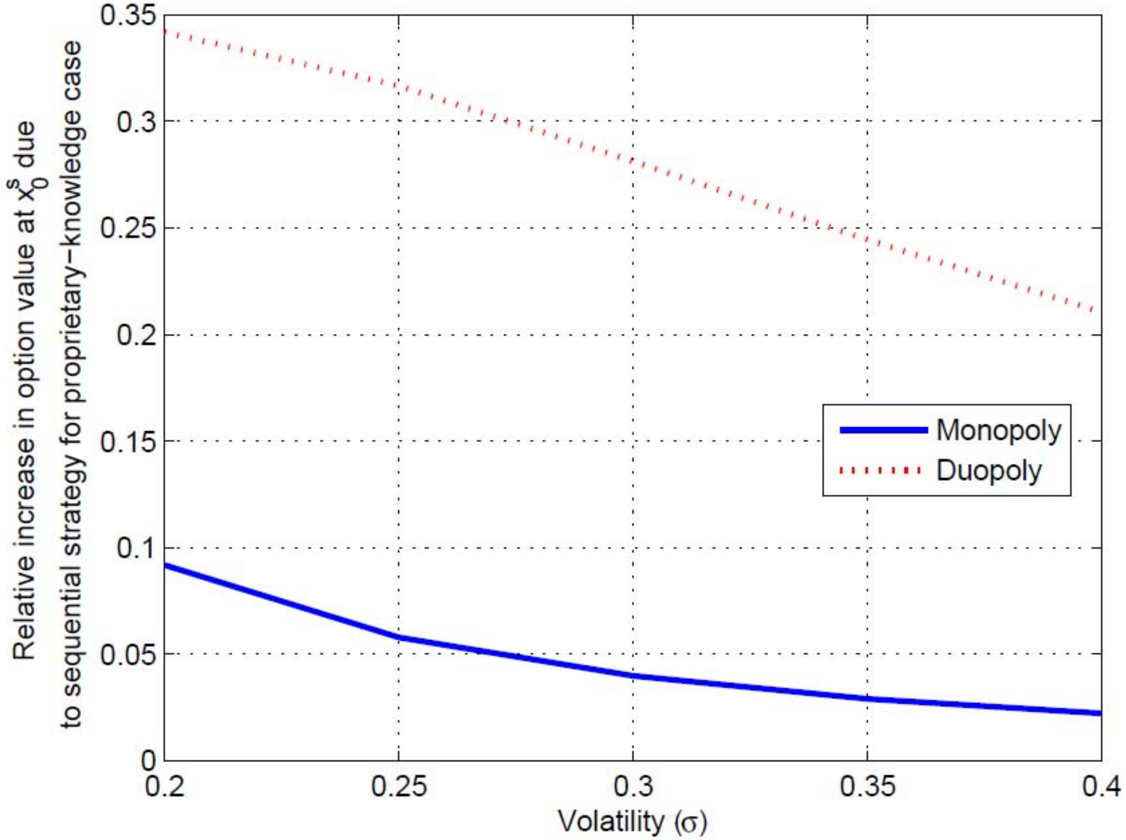
Numerical Example: Spillover Duopoly Effect of Competition with Lower First-Mover Advantage

$$\frac{V_{00}^{j,d}(x_0^s)}{V_0^d(x_0^s)} \text{ or } \frac{V_{00}^{j,s}(x_0^s)}{V_0^s(x_0^s)}$$



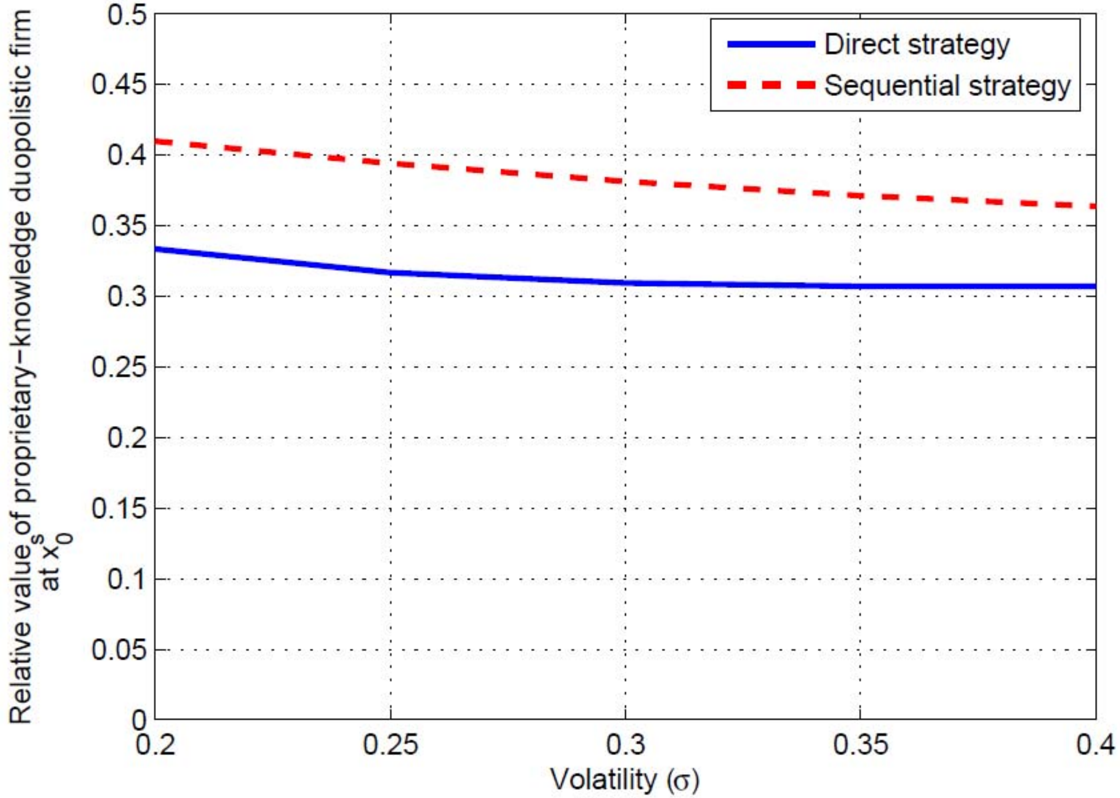
Numerical Example: Proprietary Duopoly Value of Flexibility

$$\frac{V_0^s(x_0^s)-V_0^d(x_0^s)}{V_0^d(x_0^s)}$$



Numerical Example: Proprietary Duopoly Effect of Competition

$$\frac{V_{00}^{j,d}(x_0^s)}{V_0^d(x_0^s)} \text{ or } \frac{V_{00}^{j,s}(x_0^s)}{V_0^s(x_0^s)}$$



Questions