How useful is the deterministic solution?

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Outline

The tradition

- In parts of OR there are long deterministic traditions
- Only recently have main-stream researchers started to consider uncertainty
  - Many researchers are not applied, so who cares?
  - The world is getting more uncertain?
  - The deterministic solutions simply do not work properly
  - There is now computational hope
- But even so, are deterministic solutions suddenly useless? Is there no middle way?

In the stochastoc programming camp

- The world is uncertain, so obvioulsy our models are more correct.
- But to be sure, we measure how badly things can go: *The value of the stochastic solution* – VSS.
- Measures the expected loss when using the deterministic solution.
  - If we have hard constraints, the expected cost of the deterministic solution is often ∞.
  - If we use soft constraints we can make the expected cost using the deterministic solution arbitrarily bad by setting penalties high.
- I don't think this has been manipulated, but even so ...

The overall goal and questions

- Deeper understanding of the expected value solution and its relationship to its stochastic counterpart.
- Useful because it could reveal some general properties of the underlying problem and help to predict how the stochastic model will perform when:
  1. the stochastic model is **actually solvable**, but it is solved repeatedly (daily);
  2. the stochastic model is **not even solvable**.

QUESTIONS

- Does the stochastic optimal solution **inherit** properties from the deterministic one?
- What is the reason if VSS is big?
  - **wrong choice of variables** (like which facilities to open)?
  - **wrong values** (like capacities)?
- Can we **update** the stochastic solution from the determinstic one?

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The Recourse Problem

$$RP = \min_{x \in X} E_{\boldsymbol{\xi}} z\left(x, \boldsymbol{\xi}\right)$$

Specific cases will be given later.

Tests

Test A: The **classical evaluation of the expected value solution** $\bar{x}(\bar{\xi})$. We calculate $EEV = E_{\boldsymbol{\xi}} z\left(\bar{x}\left(\bar{\xi}\right), \boldsymbol{\xi}\right)$ and compare it with RP using

$$VSS = EEV - RP$$

Test B: Fix at zero all first stage variables which are at zero in the expected value solution and then solve the stochastic program (**skeleton**).

Test C: Consider the expected value solution $\bar{x}(\bar{\xi})$ as a **starting point** to the stochastic model. Test if the expected value solution is **upgradeable**.

Test D: According to the interpretation of the variables and model, **partial information** is imported from the expected value solution.

[Thapalia, B.K., Wallace, S.W., Kaut, M. and Crainic, T.G. *Comput. Manag. Sci.* to appear.]

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Measures of the quality of the deterministic solution

Test A Let $EEV = E_{\boldsymbol{\xi}} z\left(\bar{x}\left(\bar{\xi}\right), \boldsymbol{\xi}\right)$ and compare with RP by $VSS = EEV - RP$.

Test B Let $\hat{x}$ be the solution of:

$$\begin{array}{ll} \min_{x \in X} & E_{\boldsymbol{\xi}} z\left(x, \boldsymbol{\xi}\right) \\ \text{s.t.} & x_j = \bar{x}_j(\bar{\xi}), \quad j \in \mathcal{J}. \end{array}$$

where $\mathcal{J} = \{j \mid \bar{x}_j(\bar{\xi}) = 0\}$.

We compute:

$$ESSV = E_{\boldsymbol{\xi}} z\left(\hat{x}, \boldsymbol{\xi}\right)$$ **expected skeleton solution value**

and we compare it with RP by means of:

$$VSS \geq LUSS = ESSV - RP$$ **loss using skeleton the solution**

[Maggioni, F. and Wallace, S.W. *Annals of Operations Research*, to appear.]

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Measures of the quality of the deterministic solution

Test C Let $\widetilde{x}$ be the solution of:

$$\begin{array}{ll} \min_{x \in X} & E_{\boldsymbol{\xi}} z(x, \boldsymbol{\xi}) \\ \text{s.t.} & x \geq \bar{x}(\bar{\xi}) \ . \end{array}$$

We then compute:

$$EIV = E_{\boldsymbol{\xi}} z(\widetilde{x}, \boldsymbol{\xi})$$ **expected input value**

and we compare it with RP, by means of:

$$LUDS = EIV - RP$$ **loss of upgrading the deterministic solution**

We have:

$$VSS \geq LUDS \geq 0 \ .$$

A single sink transportation problem

Sicily
- Isola d. Femmine (PA)
- Porto Empedocle (AG)

Calabria
- Castrovillari (CS)
- Vibo Valentia (VV)

[Maggioni, F., Kaut, M. and Bertazzi, L. (2009) *Comput. Manag. Sci*, **6**, 251-267.]

Description of the problem

- All the vehicles are leased from an external transportation company, which we assume to have an unlimited fleet.
- The vehicles must be booked in advance, before the demand and production capacity are revealed.
- Only full load shipments are allowed.
- When the demand and the production capacity become known, there is an option to cancel some of the reservations against a cancellation fee.
- If the quantity delivered from the four suppliers using the booked vehicles is not enough to satisfy the demand in Catania, the residual quantity is purchased from an external company at a higher price.

The variables

- X_i: The number of vehicles booked from supplier $i \in \mathcal{I}$;
- Z_i^s: The number of vehicles actually used from supplier $i \in \mathcal{I}$ in scenario $s \in \mathcal{S}$;
- Y^s: The quantity bought externally in scenario $s \in \mathcal{S}$;

The parameters

Deterministic parameters

- q: fixed **vehicle capacity**;
- g: **maximum monthly unloading capacity** of Catania's warehouse;
- l_0: **inventory level** of Catania at the beginning of the month considered;
- $l_{\max}$: the **maximum storage capacity** of Catania;
- t_i are the unitary **transportation costs** from supplier $i \in \mathcal{I}$ to the retailer;
- c_i are the unitary **production costs** of clinker for $i \in \mathcal{I}$;
- b: **buying cost** of clinker from an external firm; $b > \max_i(t_i + c_i)$, $i \in \mathcal{I}$;
- α: **canceling fee** applied for vehicles ordered but not used;

Stochastic parameters

- $p^s = 1/|S|$: the **probability** of scenario $s \in \mathcal{S}$:
- d^s: **stochastic monthly demand** in scenario $s \in \mathcal{S}$
- a_i^s: **monthly production capacity** from supplier $i \in \mathcal{I}$ to Catania in scenario $s \in S$.

$$\min q \sum_{i=1}^{\mathcal{I}} t_i X_i + \sum_{s=1}^{S} p^s \left[bY^s - (1-\alpha)\, q \sum_{i=1}^{\mathcal{I}} t_i \left(X_i - Z_i^s \right) \right] ,$$

$$\begin{aligned}
\text{such that} \qquad \sum_{i=1}^{\mathcal{I}} q X_i &\leq g \; ; \\
l_0 + q \sum_{i=1}^{\mathcal{I}} Z_i^s + Y^s - d^s &\geq 0 \, , \quad \forall \; s \in \mathcal{S}; \\
l_0 + q \sum_{i=1}^{\mathcal{I}} Z_i^s + Y^s - d^s &\leq l_{\max} \, , \quad \forall \; s \in \mathcal{S}; \\
Z_i^s &\leq X_i \, , \quad \forall \; i \in \mathcal{I} \, , \forall \; s \in \mathcal{S}; \\
q Z_i^s &\leq a_i^s \, , \quad \forall \; i \in \mathcal{I} \, , \; \forall s \in \mathcal{S}; \\
Y^s &\geq 0 \, , \quad \forall \; s \in \mathcal{S} \; ; \\
X_i &\in \mathbb{N} \, , \quad \forall \; i \in \mathcal{I}; \\
Z_i^s &\in \mathbb{N} \, , \quad \forall \; i \in \mathcal{I}, \; \forall \; s \in \mathcal{S}.
\end{aligned}$$

Test for single sink transportation problem

	AG	CS	PA	VV	Objective value (€)
deterministic	206	0	530	0	294 898=EV
stochastic	400	0	563	117	438 301=RP
Test A	206	0	530	0	495 788=EEV
Test B	400	0	637	0	462 214=ESSV
Test C	400	0	563	117	438 301=EIV=RP

- The model sorts the suppliers according to the transportation costs.

$$\text{VSS} = 495\,788 - 438\,304 = 57\,384$$

Why is the deterministic solution bad?

- too few booked vehicles from the four suppliers?
- wrong suppliers?

$$\begin{aligned} LUSS &= 23\,910 \quad \text{wrong suppliers} \\ LUDS &= 0 \quad \text{perfectly upgradeable} \end{aligned}$$

The Dakota problem

The Dakota Furniture Company wants to determine the number of desks/tables/chairs to produce and the resources to require to maximize the profit under uncertain demand.

The variables

- x_w: amounts bought of resource $w \in \mathscr{W}$;
- y_p: amounts produced of product $p \in \mathscr{P}$;
- z_p^s: amounts sold of product $p \in \mathscr{P}$ in scenario $s \in \mathscr{S}$;

The parameters

- c_w: unit cost for resource $w \in \mathscr{W}$;
- e_p: unit income for product $p \in \mathscr{P}$;
- $m_{w,p}$: amounts of resource $w \in \mathscr{W}$ neede for one utnit of product $p \in \mathscr{P}$;
- D_p^s: demand for product $p \in \mathscr{P}$ in scenario $s \in \mathscr{S}$;

[Higle, J.L. and Wallace, S.W. (2003) *Interfaces* **33**(4), 53-60.]

The Dakota problem

$$\max \; -\sum_{w\in\mathscr{W}} c_w x_w + \sum_{s\in\mathscr{S}} p^s \sum_{p\in\mathscr{P}} e_p z_p^s$$

$$\begin{aligned}
\text{s.t.} \quad -x_w + \sum_{p\in\mathscr{P}} m_{w,p} y_p &\leq 0\,, && w \in \mathscr{W} \\
z_p^k - D_p^s &\leq 0\,, && p \in \mathscr{P},\ s \in \mathscr{S} \\
z_p^s - y_p &\leq 0\,, && p \in \mathscr{P},\ s \in \mathscr{S} \\
x_w &\geq 0\,, && w \in \mathscr{W} \\
y_p &\in \mathbb{N}\,, && p \in \mathscr{P} \\
z_p^s &\in \mathbb{N}\,, && p \in \mathscr{P},\ s \in \mathscr{S}
\end{aligned}$$

Tests for Dakota furniture problem

	lumb.	carp.	fin.	desks	tables	chairs	Obj. value (€)
deterministic	1 950	850	487.5	150	125	0	4 165=EV
stochastic	1 060	420	265	50	110	0	1 142=RP
Test A	1 950	850	487.5	150	125	0	865=EEV
Test B	1 060	420	265	50	110	0	1 142=ESSV=RP
Test C	1 950	850	487.5	150	125	0	865=EEV=EIV
Test D	1 950	850	487.5	110	150	20	885

$$VSS = 1142 - 865 = 277$$

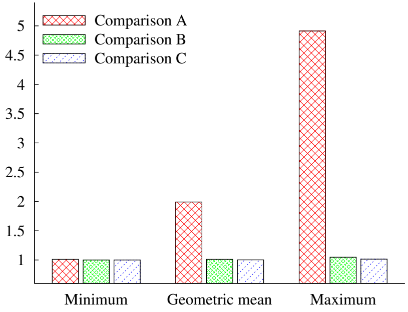
Why is the deterministic solution bad?

- wrong number of resources/items produced?
- wrong types of items produced (desks and tables instead of chairs)?

$$LUSS = 0 \qquad \text{perfect skeleton solution}$$
$$LUDS = VSS = 277 \qquad \text{no upgradability}$$

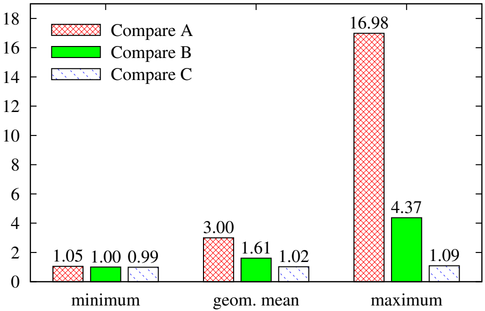
Network design – one supply node

- Single commodity case
- No time dependence (like water)
- Can solve cases up to about 15 nodes, 20 arcs, 10 random variables using CPLEX



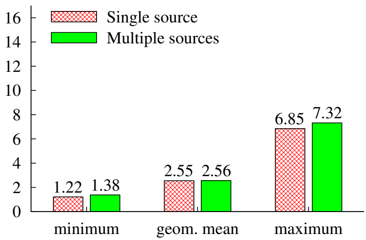
Multiple supply nodes

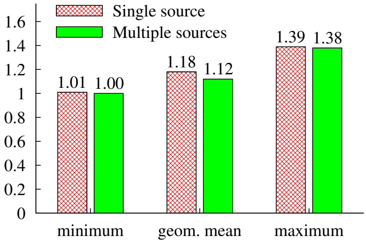
- Skelton is now not so good
- Still upgradable



Random edge capacities

Test A on the left, test B on the right





- The deterministic solution is not very good
- But the skeleton is rather good

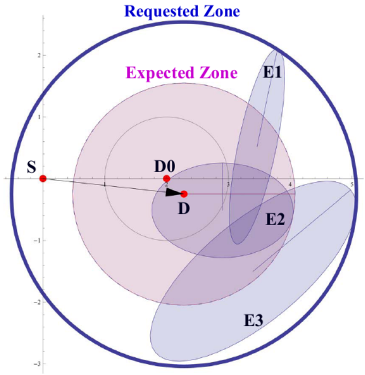
An algorithm for these problems?

- Find the skeleton by solving a *deterministic* network design problem
- Find capacities by solving a stochastic *linear* program
- This is not generally a good approach

Mobile ad-hoc network problem

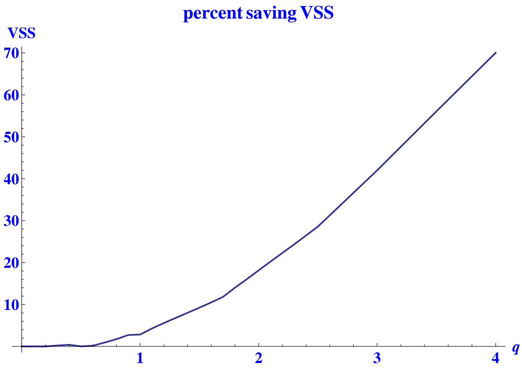
2-STAGE PROCEDURE

- calculate intial expected zone where we expect to find D;
- If D is in the expected zone, no further action is needed (see $E2$);
- If D is not found, the disk is then enlarged to get a new requested zone.



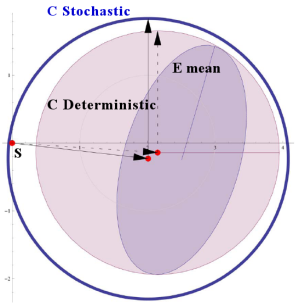
[Maggioni, Potra, Bertocchi and Allevi, (2009) *JOTA*, **143**, 309-328.]

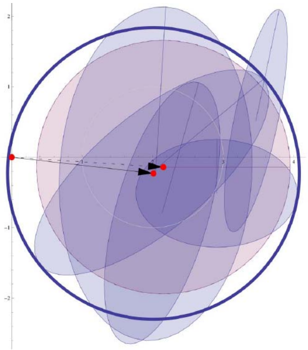
Sensitivity analysis against second stage cost of flooding: VSS



- For low q it is safe to save time by solving the *deterministic* mean value problem instead of the complex stochastic one.
- Monotonically increasing VSS: the deterministic first stage radius is not enough to contain all the scenario ellipses.

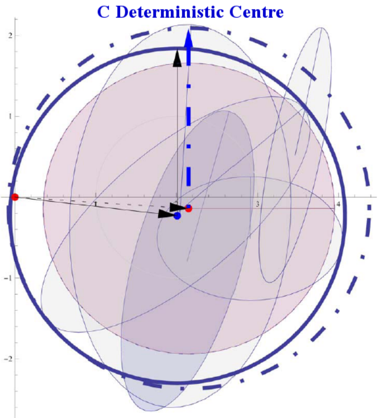
Tests for mobile ad-hoc network problem

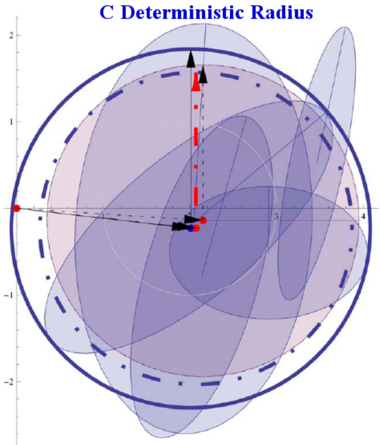




	Objective value
deterministic	5.38
stochastic	8.52
Test A	9.31
fixing det. centre	8.77
fixing det. radius	9.03

Tests for mobile ad-hoc network problem





- The deterministic model delivers a **good** choice of the **centre** but **not** of the **radius**, too small to contain larger ellipses scenarios.

Conclusions on VSS

- The **Quality** of the **expected value solution** in terms of its **structure** and **upgradeability** to the stochastic solution has been analyzed.
- Using these tests we can identify the main causes of badness/goodness of the expected value solution:
  - wrong choice of **variables** $(0 < LUSS \leq VSS)$.
  - wrong **values** $(LUSS = 0)$.
  - **non-upgradeability** of the deterministic solution $(LUDS > 0)$.
- Based on these results, the deterministic solutions can provide information and at times be part of a heuristic for the stochastic case.

Expected Value of Perfect Information – EVPI

While VSS measures the value of using a stochastic model, EVPI measures the value of knowing. The "P" stands for Perfect, but we can also measure the value of knowing "more".

So if EVPI is large, it is important to learn more.

$$VSS = E_{\boldsymbol{\xi}} z\left(\bar{x}\left(\bar{\xi}\right), \boldsymbol{\xi}\right) - RP \quad (1)$$

$$EVPI = RP - E_{\boldsymbol{\xi}} \min_{x \in X} z\left(x, \boldsymbol{\xi}\right) \quad (2)$$

A more awkward measure is the following which (sort of) applies if you can solve only deterministic models

$$E_{\boldsymbol{\xi}} z\left(\bar{x}\left(\bar{\xi}\right), \boldsymbol{\xi}\right) - E_{\boldsymbol{\xi}} \min_{x \in X} z\left(x, \boldsymbol{\xi}\right)$$

The difference

It is crucial that you pick the right measure while wanting to make a statement.

- One is about the importance of handling the uncertainty correctly while modellling
- the other about how important it is to learn more (should you invest in information collection?).