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Solving Multi-Leader-Common-Follower Games

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Abstract

Multi-leader-common-follower games arise when modeling two or more competitive firms, the leaders, that commit to their decisions prior to another group of competitive firms, the followers, that react to the decisions made by the leaders. These problems lead in a natural way to equilibrium problems with equilibrium constraints (EPECs). We develop a characterization of the solution sets for these problems and examine a variety of nonlinear optimization and nonlinear complementarity formulations of EPECs. We distinguish two broad cases: problems where the leaders can cost-differentiate and problems with price-consistent followers. We demonstrate the practical viability of our approach by solving a range of medium-sized test problems.

Keywords: Nash games, Stackelberg games, nonlinear programming, nonlinear complementarity, NCP, MPEC, equilibrium problems with equilibrium constraints.

AMS-MSC2000: 90C30, 90C33, 90C55, 49M37, 65K10.

1 Introduction

Nash games [23, 24] model competitive behavior among a set of players that make simultaneous decisions. A Nash equilibrium is a set of strategies in which each individual player has chosen an optimal strategy given the strategies chosen by the other players. A Stackelberg (*single-leader-follower*) game [31, 20] arises when one player, the *leader*, commits to a strategy, while the remaining players, the *followers*, react to the strategy selected by competing among themselves. That is, the reaction of the followers is a Nash

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equilibrium parametrized by the decision variables for the leader. The leader chooses an optimal strategy knowing how the followers will react. Between these two extremes is the *multi-leader-follower* game in which multiple competitive firms commit to their decisions prior to a number of competitive followers that react to the decisions made by the leaders. Multi-leader-follower games can be further differentiated into those in which the follower responses are constrained to be common across all leaders and those in which the followers respond differently to each leader. We consider only the former, the *multi-leader-common-follower* game. Problems of this type arise, for example, in the analysis of deregulated electricity markets [13, 4] in which some of the large energy producers are the leaders and the smaller energy producers and independent system operator are the followers. One formulation of multi-leader-common-follower games uses the novel modeling paradigm of equilibrium problems with equilibrium constraints (EPECs). This paradigm was introduced in [27] and further developed in [7] in the context of modeling competition in European electricity markets. In this paper, we characterize the solution sets for EPECs, describe practical approaches for solving them, and apply these techniques to several medium-sized problems.

Optimality conditions for EPECs are studied in [21] in the context of multiobjective optimization. Early algorithmic work on EPECs has focused on diagonalization techniques such as Gauss-Jacobi and Gauss-Seidel methods. Such methods solve a cyclic sequence of single-leader-follower games until the decision variables of all leaders reach a fixed point. In [32], Su proposes a sequential nonlinear complementarity problem (NCP) approach for solving EPECs. This approach is related to the relaxation technique used when solving mathematical programs with equilibrium constraints (MPECs) that relaxes the complementarity conditions and drives the relaxation parameter to zero [29].

In this paper, we develop several EPEC reformulations that simultaneously solve the optimization problems for all leaders by exploiting the insight gained from applying nonlinear solvers to MPECs. In particular, we derive a (nonsquare) NCP formulation of the EPEC based on the equivalence between the Karush-Kuhn-Tucker (KKT) conditions of the individual MPECs and strong stationarity. This NCP formulation is analyzed further, and we derive equivalent MPEC and nonlinear programming (NLP) formulations. We also introduce the notion of price consistency and show that, for a restricted class of multi-leader-common-follower games, the EPEC reduces to a square NCP that can be solved by applying standard NCP methods.

This paper is organized as follows. The next section briefly derives single-leader-follower games and reviews recent progress in solving them. Section 3 extends these ideas to multi-leader-common-follower games and discusses a characterization of the solution set. Section 4 introduces a new equilibrium concept and shows how equilibrium points can be computed reliably for general multi-leader-common-follower games by solving nonlinear optimization problems. Section 5 introduces an alternative price-consistent formulation that gives rise to a square nonlinear complementarity problem. This formulation, however, is valid only for a restricted class of multi-leader-common-follower games. Section 6 explores the different formulations of equilibrium points and investigates the suitability of nonlinear solvers.

2 Single-Leader-Follower Games

In this section, we briefly review pertinent properties of single-leader-follower games. These properties are extended in Section 3 when solving multi-leader-common-follower games.

The single-leader-follower game is played by a leader that commits to a decision and a number of competitive followers that react to the decision made. That is, given a strategy x for the leader, the ℓ followers choose their strategies such that

$$w_j^* \in \left\{ \begin{array}{ll} \underset{w_j \geq 0}{\text{argmin}} & b_j(x, \hat{w}_j) \\ \text{subject to} & c_j(x, w_j) \geq 0 \end{array} \right\} \quad \forall j = 1, \ldots, \ell, \tag{2.1}$$

where $\hat{w}_j = (w_1^*, \ldots, w_{j-1}^*, w_j, w_{j+1}^*, \ldots, w_\ell^*)$. Each player its own objective and constraints, which need not be the same for all players. This problem is a Nash game parametrized by the decision made by the leader. If (2.1) is convex and satisfies a constraint qualification for each follower, then the condition that each follower chooses an optimal strategy is equivalent to the parametric nonlinear complementarity problem

$$\begin{array}{rcll} 0 \leq w_j & \perp & \nabla_{w_j} b_j(x, w) - \nabla_{w_j} c_j(x, w_j) z_j \geq 0 & \forall j = 1, \ldots, \ell \\ 0 \leq z_j & \perp & c_j(x, w_j) \geq 0 & \forall j = 1, \ldots, \ell, \end{array} \tag{2.2}$$

where the complementarity condition $\perp$ means that componentwise either the left or the right inequality is active. This parametric NCP is the collection of KKT conditions for the optimization problems solved by the followers. By defining variables $y_j = (w_j, z_j)$ and functions

$$h_j(x, y_j) = \begin{pmatrix} \nabla_{w_j} b_j(x, w) - \nabla_{w_j} c(x, w_j) z_j \\ c_j(x, w_j) \end{pmatrix},$$

and by introducing slack variables s, one can write this parametric NCP as

$$\begin{array}{c} h(x, y) - s = 0 \\ 0 \leq y \perp s \geq 0, \end{array} \tag{2.3}$$

where $h(x, y) = (h_1(x, y_1), \ldots, h_\ell(x, y_\ell))$.

The leader selects a strategy by optimizing its own objective function $f(x, y)$ subject to its own set of constraints $g(x) \geq 0$ and the parametrized NCP (2.3):

$$\begin{array}{ll} \underset{x \geq 0, y, s}{\text{minimize}} & f(x,y) \\ \text{subject to} & g(x) \geq 0 \\ & h(x,y) - s = 0 \\ & 0 \leq y \ \perp \ s \geq 0. \end{array} \tag{2.4}$$

That is, the leader makes an optimal decision knowing how the competitive followers will react to that decision. Problem (2.4) is an MPEC. The left graph of Figure 1 shows an example of a single-leader-follower game arising in the analysis of electricity markets in which one large electricity producer acts as the leader, with a number of smaller producers acting as followers that play a Nash game with the independent system operator.

A number of limitations arise when writing a Nash game using complementarity constraints. If the problems solved by the followers either are nonconvex or do not satisfy a constraint qualification, then (2.1) and (2.3) are not equivalent. In fact, the solutions to (2.3) include all stationary points of (2.1); some players may be at a saddle point, a local maximizer, or a local minimizer that is not a global minimizer for their optimization problem at solutions to (2.3). Thus, a solution to (2.4) may not correspond to a solution to the single-leader-follower game in these situations. Moreover, if a constraint qualification is not satisfied by each optimization problem in (2.1), then (2.3) may not have a solution because the multipliers z_j may not exist, even though (2.1) can have a solution. We accept these limitations because it is not clear at present how they can be avoided in practice.

The formulation presented does not allow the constraints of each follower to depend on the strategy chosen by the other followers so that the standard notion of a Nash game is recovered. Generalized Nash and Stackelberg games are obtained when the constraints are allowed to depend on the choices made by the other players. Thus, $c_j(x, w_j)$ becomes $c_j(x, w)$, and $g(x)$ can also depend on the responses becoming $g(x, y)$. In this case, the MPEC (2.4) becomes

$$\begin{array}{ll} \underset{x \geq 0, y, s}{\text{minimize}} & f(x,y) \\ \text{subject to} & g(x,y) \geq 0 \\ & h(x,y) - s = 0 \\ & 0 \leq y \ \perp \ s \geq 0, \end{array} \tag{2.5}$$

where h is now defined to take $c_j(x, w)$ into account. Difficulties associated with the solutions to generalized Nash games are discussed in [27]. Some test problems in Section 6 are derived from generalized Nash and Stackelberg games.

One attractive solution approach to (2.4) or (2.5) is to replace the complementarity condition by a nonlinear inequality, such as $y^T s \leq 0$ or $Ys \leq 0$, where Y is the diagonal

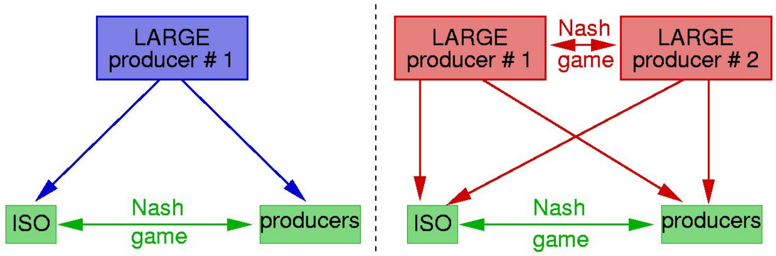

Figure 1: Structure of single-leader-follower games and multi-leader-common-follower games

matrix with y along its diagonal. This equivalent nonlinear program can then be solved by applying standard NLP solvers. Unfortunately, this NLP violates the Mangasarian-Fromovitz constraint qualification (MFCQ) at *any* feasible point [29]. This failure of MFCQ implies that the multiplier set is unbounded, the central path fails to exist, the active constraint normals are linearly dependent, and linearizations can become inconsistent *arbitrarily close* to a solution [11]. In addition, early numerical experience with this approach was disappointing [2]. As a consequence, solving MPECs as NLPs has been commonly regarded as numerically unsafe.

The failure of MFCQ in the equivalent NLP can be traced to the formulation of the complementarity constraint as $Ys \leq 0$. Consequently, algorithmic approaches have focused on avoiding this formulation. Instead, researchers have developed special-purpose algorithms for MPECs, such as branch-and-bound methods [2], implicit nonsmooth approaches [25], piecewise SQP methods [19], and perturbation and penalization approaches [6] analyzed in [30]. All of these techniques, however, require significantly more work than a standard NLP approach.

Recently, researchers have shown that MPECs can be solved reliably and efficiently by replacing the complementarity constraint with $Ys \leq 0$ and applying standard nonlinear optimization solvers [1, 3, 10, 11, 16, 17, 18, 28]. The key observation in proving convergence of such an approach is that strong stationarity [29] is equivalent to the KKT conditions of the equivalent NLP. The notion of strong stationarity is presented in the context of the general MPEC (2.5) and the NLP reformulation

$$\begin{array}{ll} \underset{x\geq 0,y,s}{\text{minimize}} & f(x,y) \\ \text{subject to} & g(x,y) \geq 0 \\ & h(x,y) - s = 0 \\ & y \geq 0,\ s \geq 0,\ Ys \leq 0. \end{array} \tag{2.6}$$

Definition 2.1 *A point (x, y, s) is called a strongly stationary point of the MPEC (2.5) if and only if there exist multipliers $\chi \geq 0, \lambda \geq 0, \mu, \psi, \sigma$ such that*

$$
\begin{aligned}
\nabla_x f(x,y) - \nabla_x g(x,y)\lambda - \nabla_x h(x,y)\mu - \chi &= 0 \\
\nabla_y f(x,y) - \nabla_y g(x,y)\lambda - \nabla_y h(x,y)\mu - \psi &= 0 \\
\mu - \sigma &= 0 \\
g(x,y) &\geq 0 \\
h(x,y) - s &= 0 \\
x, y, s &\geq 0 \\
x^T\chi = 0 \;\; \text{and} \;\; g(x,y)^T\lambda = 0 \;\; \text{and} \;\; y^T\psi = 0 \;\; \text{and} \;\; s^T\sigma &= 0 \\
\text{if} \;\; y_i = s_i = 0 \;\; \text{then} \;\; \psi_i, \; \sigma_i &\geq 0.
\end{aligned}
\tag{2.7}
$$

The multipliers on the simple bounds ψ_i and σ_i are nonnegative only if $y_i = s_i = 0$, as is consistent with the observation that $y_i > 0$ implies $s_i = 0$, and σ_i is therefore the multiplier on an equality constraint whose sign is not restricted.

Fletcher et al. [11] have shown that strong stationarity is equivalent to the KKT conditions of the equivalent NLP (2.6). That is, there exist multipliers $\chi \geq 0, \lambda \geq 0, \mu$, $\psi \geq 0, \sigma \geq 0, \xi \geq 0$ such that

$$
\begin{aligned}
\nabla_x f(x,y) - \nabla_x g(x,y)\lambda - \nabla_x h(x,y)\mu - \chi &= 0 \\
\nabla_y f(x,y) - \nabla_y g(x,y)\lambda_y - \nabla h(x,y)\mu - \psi + S\xi &= 0 \\
\mu - \sigma + Y\xi &= 0 \\
0 \leq g(x,y) \;\perp\; \lambda &\geq 0 \\
h(x,y) - s &= 0 \\
0 \leq x \;\perp\; \chi &\geq 0 \\
0 \leq y \;\perp\; \psi &\geq 0 \\
0 \leq s \;\perp\; \sigma &\geq 0 \\
0 \leq -Ys \;\perp\; \xi &\geq 0.
\end{aligned}
\tag{2.8}
$$

These strong stationarity conditions are used in Section 3 to derive nonlinear formulations of EPECs.

When an MPEC constraint qualification is satisfied, multipliers for the equivalent NLP (2.6) exist and form a ray. Moreover, SQP methods converge to a minimum norm multiplier corresponding to the base of the ray [11]. The aim of this paper is to demonstrate that strong stationarity can be applied within the context of multi-leader-common-follower games to define equilibrium points, thus making EPECs amenable to approaches based on nonlinear optimization or nonlinear complementarity.

We conclude this section by recalling the definition of an MPEC constraint qualification. This constraint qualification will be used when constructing equilibrium conditions for multi-leader-common-follower games.

Definition 2.2 *The MPEC (2.5) satisfies an MPEC linear independence constraint qualification (MPEC-LICQ) if the NLP (2.6) without the complementarity condition $Ys \le 0$ satisfies an LICQ.*

3 Multi-Leader-Common-Follower Games

Multi-leader-common-follower games arise when two or more competitive leaders commit to decisions and the common competitive followers react to the decisions made. The right graph of Figure 1 shows an example of a multi-leader-common-follower game in which two large electricity producers act as leaders, with a number of smaller producers acting as followers that play a Nash game with the independent system operator. Such games can be modeled as equilibrium problems with equilibrium constraints. The aim is to find an equilibrium point where no leader can improve its objective given the strategies chosen by the other leaders and the reaction of the followers. The followers compute an equilibrium point where no follower can improve its objective given the strategies committed to by the leaders and those selected by the other followers. This goal is achieved by extending strong stationarity (Definition 2.1) to equilibrium problems with equilibrium constraints.

Let $k > 1$ be the number of leaders, and denote by $x_i,\ i = 1, \ldots, k$, the decision variables for leader i. The leader variables are abbreviated by $x = (x_1, \ldots, x_k)$. The optimization problem solved by leader i gives rise to the following MPEC:

$$\begin{array}{ll} \underset{x_i \ge 0, y, s}{\text{minimize}} & f_i(\hat{x}_i, y) \\ \text{subject to} & g_i(\hat{x}_i, y) \ge 0 \\ & h(\hat{x}_i, y) - s = 0 \\ & 0 \le y \ \perp\ s \ge 0, \end{array}$$

where $\hat{x}_i = (x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_k^*)$. The multi-leader-common-follower game is then defined as a solution to the following collection of MPECs:

$$(x_i^*, y^*, s^*) \in \left\{ \begin{array}{ll} \underset{x_i \ge 0, y, s}{\text{argmin}} & f_i(\hat{x}_i, y) \\ \text{subject to} & g_i(\hat{x}_i, y) \ge 0 \\ & h(\hat{x}_i, y) - s = 0 \\ & 0 \le y \ \perp\ s \ge 0 \end{array} \right\} \ \forall i = 1, \ldots, k. \tag{3.1}$$

Asserting the existence and uniqueness of a solution to the multi-leader-common-follower game is difficult because each optimization problem solved is nonconvex. A further complication is the common decision variables, (y^*, s^*), which appear in the optimization problem solved by each leader. The solution set, however, can be characterized by the intersection of the solution sets for a series of related generalized Nash games.

In particular, let $\mathcal{S}$ denote the set of Nash equilibria for the multi-leader-common-follower game (3.1). This solution set includes only the primal variables and does not include multipliers on the constraints. Associated with this game, one can construct k related games, where the common decision reached by the followers is attached to exactly one leader. For player $\bar{i} \in \{1, \ldots, k\}$, this game has the following form:

$$(x^*_{\bar{i}}, y^*, s^*) \in \left\{ \begin{array}{ll} \displaystyle\operatorname*{argmin}_{x_{\bar{i}} \geq 0, y, s} & f_{\bar{i}}(\hat{x}_{\bar{i}}, y) \\ \text{subject to} & g_{\bar{i}}(\hat{x}_{\bar{i}}, y) \geq 0 \\ & h(\hat{x}_{\bar{i}}, y) = s \\ & 0 \leq y \perp s \geq 0 \end{array} \right\} \tag{3.2a}$$

$$x^*_i \in \left\{ \begin{array}{ll} \displaystyle\operatorname*{argmin}_{x_i \geq 0} & f_i(\hat{x}_i, y^*) \\ \text{subject to} & g_i(\hat{x}_i, y^*) \geq 0 \\ & h(\hat{x}_i, y^*) = s^* \end{array} \right\} \forall i = 1, \ldots, k, i \neq \bar{i}. \tag{3.2b}$$

Let $\mathcal{S}_{\bar{i}}$ denote the set of Nash equilibria for each modified game (3.2). Again, this solution set includes only the primal variables and does not include multipliers on the constraints. Then, we have the following characterization of the Nash equilibria to multi-leader-common-follower games.

Theorem 3.1 *Let $\mathcal{S}$ denote the set of Nash equilibria for the multi-leader-common-follower game (3.1), and let $\mathcal{S}_{\bar{i}}$ denote the set of Nash equilibria for (3.2). Then,*

$$\mathcal{S} = \cap_{\bar{i}=1}^{k} \mathcal{S}_{\bar{i}}. \tag{3.3}$$

Proof. Select a point in (x^*, y^*, s^*) in $\mathcal{S}$. For each $\bar{i} = 1, \ldots, k$, (x^*_i, y^*, s^*) solves the optimization problem in (3.1) for player $\bar{i}$. Therefore, (3.2a) also holds for player $\bar{i}$. One can then show that for each $i = 1, \ldots, k$, x^*_i is a solution to (3.2b). In particular, if it were possible to obtain a lower objective function value for this problem, then (x^*_i, y^*, s^*) would not be optimal for (3.1). Hence, $(x^*, y^*, s^*) \in \cap_{\bar{i}=1}^{k} \mathcal{S}_{\bar{i}}$.

For the opposite, select a point (x^*, y^*, s^*) in $\cap_{\bar{i}=1}^{k} \mathcal{S}_{\bar{i}}$. This point is optimal for each optimization problem in (3.1) because of the intersection. Hence, $(x^*, y^*, s^*) \in \mathcal{S}$. $\square$

Similar characterizations can be obtained if we seek only local solutions or strongly stationary points for each optimization problem in (3.1).

Each Nash subgame still contains a nonconvex optimization problem. However, the nonconvexity appears in only one optimization problem. Therefore, it may be easier to prove the existence of a solution to each Nash subgame. The difficulty then becomes establishing that the intersection of the solution sets is nonempty. The following result is obtained for two special cases.

Corollary 3.2 *If (3.2) has a unique Nash equilibrium for some $\bar{i}$, then the multi-leader-common-follower game (3.1) has either no Nash equilibrium or a unique Nash equilibrium. Moreover, if (3.2) has no Nash equilibrium for some $\bar{i}$, then the multi-leader-common-follower game (3.1) also has no Nash equilibrium.*

Proof. The proof follows from the intersection characterization of solutions to multi-leader-common-follower games in Theorem 3.1. □

Note that a square nonlinear complementarity problem is obtained by writing down the strong stationarity conditions for each Nash subgame. Algorithms for solving nonlinear complementarity problems can then be applied to solve the Nash subgame. For this characterization to be computationally tractable, however, one would need to compute all solutions for each Nash subgame and then take the intersection. Computing all solutions to a Nash game is not currently possible for moderately sized problems. Moreover, this result give no indications that multipliers exist on the constraints or that they are unique.

4 NCP and NLP Formulations

One computationally attractive way to solve the multi-leader-common-follower game (3.1) is to follow the same formalism as in the derivation of the complementarity problem (2.2) for Nash games. Formally, we replace each MPEC in (3.1) by its strong stationarity conditions and concatenate the equivalent KKT conditions (2.8) for all leaders $i = 1, \dots, k$. Note that the optimization problems are nonconvex because of the presence of complementarity constraints. Hence, we can compute only strongly stationary points for each optimization problem with this method. This approach formulates the multi-leader-common follower game as a (nonsquare) nonlinear complementarity problem. A range of other formulations as nonlinear programming problems are then derived. We start by constructing the NCP formulations.

4.1 NCP Formulations of Multi-Leader-Common-Follower Games

The concatenation of the strong stationarity conditions for each leader produces the following NCP formulation of the multi-leader-common-follower game (3.1).

$$\nabla_{x_i} f_i(x,y) - \nabla_{x_i} g_i(x,y)\lambda_i - \nabla_{x_i} h(x,y)\mu_i - \chi_i = 0 \quad \forall i = 1,\ldots,k \tag{4.1a}$$
$$\nabla_y f_i(x,y) - \nabla_y g_i(x,y)\lambda_i - \nabla_y h(x,y)\mu_i - \psi_i + S\xi_i = 0 \quad \forall i = 1,\ldots,k \tag{4.1b}$$
$$\mu_i - \sigma_i + Y\xi_i = 0 \quad \forall i = 1,\ldots,k \tag{4.1c}$$
$$0 \le g_i(x,y) \ \perp \ \lambda_i \ge 0 \quad \forall i = 1,\ldots,k \tag{4.1d}$$
$$h(x,y) - s = 0 \tag{4.1e}$$
$$0 \le x_i \ \perp \ \chi_i \ge 0 \quad \forall i = 1,\ldots,k \tag{4.1f}$$
$$0 \le y \ \perp \ \psi_i \ge 0 \quad \forall i = 1,\ldots,k \tag{4.1g}$$
$$0 \le s \ \perp \ \sigma_i \ge 0 \quad \forall i = 1,\ldots,k \tag{4.1h}$$
$$0 \le -Ys \ \perp \ \xi_i \ge 0 \quad \forall i = 1,\ldots,k \tag{4.1i}$$

The multipliers χ_i, ψ_i, and σ_i can be eliminated to produce a reduced, but equivalent, model. Note that the multipliers on the followers' constraints, $(\psi_i, \sigma_i, \xi_i)$, can have different values for each leader. This formulation corresponds to a scenario in which the cost of the followers' actions can be different for each leader. That is, the leaders *cost differentiate*. In contrast, in Section 5, we discuss conditions on the structure of the game (3.1) that allow us to enforce a *price-consistent formulation* in which all the multipliers $(\mu_i, \psi_i, \sigma_i, \xi_i)$ are the same for every leader.

Formally, this approach to EPECs is analogous to the formulation of the Nash game (2.1) as the complementarity problem (2.3). Unlike a Nash game, however, the MPEC (2.4) is always nonconvex because of the presence of complementarity constraints. Therefore, no simple equivalence exists between the solution set of the NCP defined by (4.1) and the solution set of the multi-leader-common-follower game (3.1). Moreover, this NCP is typically not square because equations (4.1a), (4.1b), (4.1c), and (4.1e) cannot be uniquely matched to the free variables x_i, y, s, and μ_i. Hence, this problem is harder to solve than standard Nash games.

In [32], (4.1) is solved by adding a smoothing parameter to the original complementarity condition, replacing (4.1i) by

$$-te \le -Ys \ \perp \ \xi_i \ge 0,$$

where e is a vector of ones and $t \searrow 0$. In contrast, we attack (4.1) directly by exploiting recent advances in nonlinear solvers for MPECs.

We can simplify (4.1) by noting that the complementarity condition $Ys \le 0$ in (4.1i) is always active because y and s are nonnegative. Moreover, the constraint $Ys \le 0$ can be replaced by $0 \le y \;\perp\; s \ge 0$. This formulation has the advantage that it makes the complementarity constraint transparent for the nonlinear solver, allowing, for example, different techniques to deal with the complementarity condition. Thus, we can replace the constraints (4.1g)–(4.1i) by the following set of inequalities:

$$\left.\begin{array}{rcr} 0 \le y & \perp & \psi_i \ge 0 \\ 0 \le s & \perp & \sigma_i \ge 0 \\ 0 \le y & \perp & s \ge 0 \\ & & \xi_i \ge 0 \end{array}\right\} \forall i = 1, \ldots, k. \tag{4.2}$$

An alternative formulation is to replace (4.1g)–(4.1i) with the equivalent set of conditions:

$$\left.\begin{array}{rcr} 0 \le \psi_i + s & \perp & y \ge 0 \\ 0 \le \sigma_i + y & \perp & s \ge 0 \\ & & \psi_i, \sigma_i, \xi_i \ge 0 \end{array}\right\} \forall i = 1, \ldots, k. \tag{4.3}$$

Another source of degeneracy is the fact that the multipliers form a ray and are therefore not unique. This redundancy can be removed by adding a complementarity constraint that forces the multipliers for each leader to be basic (and therefore unique), such as

$$0 \le \psi_i + \sigma_i \;\perp\; \xi_i \ge 0.$$

Combining these observations, we obtain the equivalent conditions:

$$\left.\begin{array}{rcr} 0 \le \psi_i + s & \perp & y \ge 0 \\ 0 \le \sigma_i + y & \perp & s \ge 0 \\ 0 \le \psi_i + \sigma_i & \perp & \xi_i \ge 0 \end{array}\right\} \forall i = 1, \ldots, k. \tag{4.4}$$

We now arrive at the following NCP formulation of a multi-leader-common-follower game.

$$\nabla_{x_i} f_i(x,y) - \nabla_{x_i} g_i(x,y)\lambda_i - \nabla_{x_i} h(x,y)\mu_i - \chi_i = 0 \quad \forall i = 1, \ldots, k \tag{4.5a}$$

$$\nabla_y f_i(x,y) - \nabla_y g_i(x,y)\lambda_i - \nabla_y h(x,y)\mu_i - \psi_i + S\xi_i = 0 \quad \forall i = 1, \ldots, k \tag{4.5b}$$

$$\mu_i - \sigma_i + Y\xi_i = 0 \quad \forall i = 1, \ldots, k \tag{4.5c}$$

$$0 \le g_i(x,y) \;\perp\; \lambda_i \ge 0 \quad \forall i = 1, \ldots, k \tag{4.5d}$$

$$h(x,y) - s = 0 \tag{4.5e}$$

$$0 \le x_i \;\perp\; \chi_i \ge 0 \quad \forall i = 1, \ldots, k \tag{4.5f}$$

$$0 \le \psi_i + s \;\perp\; y \ge 0 \quad \forall i = 1, \ldots, k \tag{4.5g}$$

$$0 \le \sigma_i + y \;\perp\; s \ge 0 \quad \forall i = 1, \ldots, k \tag{4.5h}$$

$$0 \le \psi_i + \sigma_i \;\perp\; \xi_i \ge 0 \quad \forall i = 1, \ldots, k. \tag{4.5i}$$

This formulation is not a square NCP because y and s are matched with multiple inequalities in (4.5g) and (4.5h), respectively.

The derivation in this section motivates the following definition. A similar definition regarding (4.1) can be found in [32].

Definition 4.1 *A solution of (3.1) is called an equilibrium point of the multi-leader-common-follower game. A solution* $(x^*, y^*, s^*, \chi^*, \lambda^*, \mu^*, \psi^*, \sigma^*, \xi^*)$ *of (4.1) or (4.5) is called a strongly stationary point of the multi-leader-common-follower game (3.1).*

The following proposition shows that equilibrium points are strongly stationary provided an MPEC-LICQ holds.

Proposition 4.1 *If* (x^*, y^*, s^*) *is an equilibrium point of (3.1) and if every MPEC of (3.1) satisfies an MPEC-LICQ, then there exist multipliers* $(\chi^*, \lambda^*, \mu^*, \psi^*, \sigma^*, \xi^*)$ *such that (4.1) and (4.5) hold.*

Proof. The statement follows directly from [29] and can also be found in [32]. If MPEC-LICQ holds, then there exist multipliers for the optimization problem solved by every leader, and (4.1) and (4.5) follow. □

We note that both (4.1) and (4.5) are degenerate in the sense that the constraints violate any constraint qualification because of the presence of complementarity conditions. In addition, the Jacobian is singular whenever any component of both y and s is zero. This fact makes it difficult to tackle (4.5) with standard NCP solvers. In the next section we derive more robust formulations of the NCP that resolve the redundancy in (4.1) and can be solved by using standard nonlinear optimization techniques.

4.2 NLP Formulations of Multi-Leader-Common-Follower Games

The redundancy inherent in the NCP formulation (4.1) can be exploited to derive nonlinear programming formulations of the multi-leader-common-follower game. This section introduces two other formulations of the NCP (4.1). The first formulation is based on the idea of forcing the EPEC to identify the basic or minimal multiplier for each leader. This formulation results in an MPEC. The second formulation penalizes the complementarity constraints and results in a well-behaved nonlinear optimization problem.

One difficulty with the NCP (4.1) is the existence of an infinite number of multipliers. Since the multipliers form a ray, however, there exists a minimum norm multiplier [11].

The first reformulation aims to find this particular multiplier by minimizing the ℓ_1-norm of the multipliers on the complementarity constraints, giving rise to the following MPEC.

$$
\begin{array}{lll}
\underset{x,y,s,\lambda,\mu,\psi,\sigma,\xi}{\text{minimize}} & \displaystyle\sum_{i=1}^{k} e^T \xi_i & \\
\text{subject to} & \nabla_{x_i} f_i(x,y) - \nabla_{x_i} g_i(x,y)\lambda_i - \nabla_{x_i} h(x,y)\mu_i - \chi_i = 0 & \forall i = 1,\dots,k \\
& \nabla_y f_i(x,y) - \nabla_y g_i(x,y)\lambda_i - \nabla_y h(x,y)\mu_i - \psi_i + S\xi_i = 0 & \forall i = 1,\dots,k \\
& \mu_i - \sigma_i + Y\xi_i = 0 & \forall i = 1,\dots,k \\
& 0 \le g_i(x,y) \;\perp\; \lambda_i \ge 0 & \forall i = 1,\dots,k \\
& h(x,y) - s = 0 & \\
& 0 \le x_i \;\perp\; \chi_i \ge 0 & \forall i = 1,\dots,k \\
& 0 \le y \;\perp\; \psi_i \ge 0 & \forall i = 1,\dots,k \\
& 0 \le s \;\perp\; \sigma_i \ge 0 & \forall i = 1,\dots,k \\
& 0 \le y \;\perp\; s \ge 0 & \\
& \xi_i \ge 0 & \forall i = 1,\dots,k
\end{array}
\tag{4.6}
$$

This reformulation violates standard constraint qualifications because it is an MPEC. However, recent developments show that MPECs like (4.6) can be solved reliably and efficiently by using standard NLP solvers [1, 10, 11]. We could also use the alternative reformulations of the complementarity constraints in (4.3) and (4.4) to obtain an MPEC.

The next formulation aims to avoid this difficulty by minimizing the complementarity constraints. This formulation of the multi-leader-common-follower game follows an idea of Moré [22] and minimizes the complementarity conditions in (4.1d) and (4.1f)–(4.5i). After introducing slacks t_i to $g_i(x,y) \ge 0$, one can write this problem as follows.

$$
\begin{array}{lll}
\underset{x,y,\nu,\mu,\xi}{\text{minimize}} & C_{pen} \;:=\; \displaystyle\sum_{i=1}^{k} \left(x_i^T \chi_i + t_i^T \lambda_i + y^T \psi_i + s^T \sigma_i \right) + y^T s & \\
\text{subject to} & \nabla_{x_i} f_i(x,y) - \nabla_{x_i} g_i(x,y)\lambda_i - \nabla_{x_i} h(x,y)\mu_i - \chi_i = 0 & \forall i = 1,\dots,k \\
& \nabla_y f_i(x,y) - \nabla_y g_i(x,y)\lambda_y - \nabla_i h(x,y)\mu_i - \psi_i + S\xi_i = 0 & \forall i = 1,\dots,k \\
& \mu_i - \sigma_i + Y\xi_i = 0 & \forall i = 1,\dots,k \\
& g_i(x,y) = t_i & \forall i = 1,\dots,k \\
& h(x,y) = s & \\
& y \ge 0,\; s \ge 0 & \\
& \chi_i \ge 0,\; \lambda_i \ge 0,\; \psi_i \ge 0,\; \sigma_i \ge 0,\; \xi_i \ge 0 & \forall i = 1,\dots,k \\
& x_i \ge 0,\; t_i \ge 0 & \forall i = 1,\dots,k
\end{array}
\tag{4.7}
$$

In this problem, the complementarity conditions have been moved into the objective by a penalty approach, and the remaining constraints are well behaved. A penalty parameter of one is always adequate because the multi-leader-common-follower game has no objective function. We could also use the other reformulations of the complementarity constraints in (4.3) and (4.4) when deriving an NLP.

The following theorem summarizes the properties of the formulations introduced in this section.

Theorem 4.1 *If $(x^*, y^*, s^*, t^*, \chi^*, \lambda^*, \mu^*, \psi^*, \sigma^*, \xi^*)$ is a local solution of (4.6), then it follows that (x^*, y^*, s^*) is a strongly stationary point of the multi-leader-common-follower game (3.1). If $(x^*, y^*, s^*, t^*, \chi^*, \lambda^*, \mu^*, \psi^*, \sigma^*, \xi^*)$ is a local solution of (4.7) with $C_{pen} = 0$, then it follows that (x^*, y^*, s^*) is a strongly stationary point of the multi-leader-common-follower game (3.1).*

Proof. The proof follows directly from the developments above. □

5 Price-Consistent Formulations

The NCP and NLP formulations of multi-leader-common-follower games can be applied to any game satisfying the assumptions in Proposition 4.1. To reduce the number of variables and constraints in these formulations, which may make the problem more tractable, one can make a price-consistency assumption. This technique restricts the solutions considered to those for which the multipliers (prices) on the common constraints are the same. Hence, the price-consistent problem may have no solution, while (4.1) has a solution. Any solution to the price-consistent problem, however, is also a solution (4.1). By further restricting the type of multi-leader-common-follower games considered, more reductions in the number of variables and constraints can be realized. In the most restrictive cases, a square NCP or standard MPEC results.

The derivation of the price-consistent problem begins by constructing a standard Nash game with three types of players: the leaders, the followers, and markets. The markets set the prices on the resource constraints. In particular, the strong stationarity conditions for the multi-leader-common-follower game (4.1) correspond exactly to Nash equilibria for the following game:

$$
\begin{array}{lll}
x_i^* & \in & \left\{ \begin{array}{ll} \underset{x_i \geq 0}{\operatorname{argmin}} & f_i(\hat{x}_i, y^*) - (\lambda_i^*)^T g_i(\hat{x}, y^*) - (\mu_i^*)^T (h(\hat{x}_i, y^*) - s^*) \end{array} \right\} \\
& & \forall i = 1, \ldots, k \\
(y^*, s^*) & \in & \left\{ \begin{array}{ll} \underset{y \geq 0, s \geq 0}{\operatorname{argmin}} & f_i(x^*, y) - (\lambda_i^*)^T g_i(x^*, y) - (\mu_i^*)^T (h(x^*, y) - s) + \sigma_i^* s^T y \end{array} \right\} \\
& & \forall i = 1, \ldots, k \\
(\lambda^*, \mu^*, \sigma^*) & \in & \left\{ \begin{array}{ll} \underset{\lambda \geq 0, \mu, \sigma \geq 0}{\operatorname{argmin}} & \sum_{i=1}^{k} \left(\lambda_i^T g_i(x^*, y^*) + \mu_i^T (h(x^*, y^*) - s^*) - \sigma_i (s^*)^T y^* \right) \end{array} \right\}.
\end{array}
\tag{5.1}
$$

Hence, any solution to (5.1) is a solution to the strong stationarity conditions for the original multi-leader-common-follower game (4.1). This problem is not a standard Nash game because y^* and s^* must be optimal for multiple (nonconvex) optimization problems. Note that as in Section 3, the solutions to (5.1) can be characterized by the intersection of the solutions for k Nash games.

The markets differentiate the resource prices for each player in (5.1). If we assume the multipliers on the common constraints are the same, that is, the prices are set by an independent entity that cannot price discriminate, then we need only one set of prices for these constraints, rather than one set of prices for each follower. This restriction leads to the following game:

$$
\begin{array}{lll}
x_i^* & \in & \left\{ \begin{array}{ll} \underset{x_i \geq 0}{\operatorname{argmin}} & f_i(\hat{x}_i, y^*) - (\lambda_i^*)^T g_i(\hat{x}_i, y^*) - (\mu^*)^T (h(\hat{x}_i, y^*) - s^*) \end{array} \right\} \\
& & \forall i = 1, \ldots, k \\
(y^*, s^*) & \in & \left\{ \begin{array}{ll} \underset{y \geq 0, s \geq 0}{\operatorname{argmin}} & f_i(x^*, y) - (\lambda_i^*)^T g_i(x^*, y) - (\mu^*)^T (h(x^*, y) - s) + \sigma^* s^T y \end{array} \right\} \\
& & \forall i = 1, \ldots, k \\
(\lambda^*, \mu^*, \sigma^*) & \in & \left\{ \begin{array}{ll} \underset{\lambda \geq 0, \mu, \sigma \geq 0}{\operatorname{argmin}} & \sum_{i=1}^{k} \lambda_i^T g_i(x^*, y^*) + \mu^T (h(x^*, y^*) - s^*) - \sigma (s^*)^T y^* \end{array} \right\}.
\end{array}
\tag{5.2}
$$

We have now reduced the size of the problem by eliminating a large number of multipliers (μ, σ) on the common complementarity constraints. To reduce the problem further, we need to make an additional assumption on the class of games considered:

[A1] The general constraints for each leader i consist of a set of constraints independent of other decision variables and a set of constraints common across all players. That is,

$$
g_i(x, y) = \begin{bmatrix} \bar{g}_i(x_i) \\ \tilde{g}(x, y) \end{bmatrix}
$$

for some functions $\bar{g}_i(x_i)$ and $\tilde{g}(x, y)$.

This assumption is not very restrictive and is commonly satisfied by the problems of interest. If Assumption [**A1**] is not satisfied, then we can solve reduced versions of (4.1), (4.5), (4.6), and (4.7) with common multipliers (μ, σ). With this assumption, we can impose a consistent set of prices on the general constraints to reduce the number of multipliers (λ) leading to the price-consistent multi-leader-common-follower game:

$$
\begin{array}{lll}
x_i^* & \in & \left\{ \begin{array}{ll} \underset{x_i \geq 0, \bar{g}_i(x_i) \geq 0}{\operatorname{argmin}} & f_i(\hat{x}_i, y^*) - (\lambda^*)^T \tilde{g}(\hat{x}_i, y^*) - (\mu^*)^T (h(\hat{x}_i, y^*) - s^*) \end{array} \right\} \\
 & & \forall i = 1, \ldots, k \\
(y^*, s^*) & \in & \left\{ \begin{array}{ll} \underset{y \geq 0, s \geq 0}{\operatorname{argmin}} & f_i(x^*, y) - (\lambda^*)^T \tilde{g}(x^*, y) - (\mu^*)^T (h(x^*, y) - s) + \sigma^* s^T y \end{array} \right\} \\
 & & \forall i = 1, \ldots, k \\
(\lambda^*, \mu^*, \sigma^*) & \in & \left\{ \begin{array}{ll} \underset{\lambda \geq 0, \mu, \sigma \geq 0}{\operatorname{argmin}} & \lambda^T \tilde{g}(x^*, y^*) + \mu^T (h(x^*, y^*) - s^*) - \sigma (s^*)^T y^* \end{array} \right\}.
\end{array}
\tag{5.3}
$$

Note that the independent constraints $\bar{g}_i(x_i) \geq 0$ have been included in the optimization problem for leader i. As before, this problem is not a standard Nash game, because of the presence of the y and s variables that must be simultaneously optimal for several optimization problems. We can, however, construct k related Nash games in which the solution set to (5.3) is the intersection of the solution sets to the related Nash games.

To produce a more standard Nash game, we need to make a rather restrictive assumption on the form of the objective function.

[**A2**] The objective function for leader i consists of a term separable in x and y and a term common across all leaders. That is,

$$f_i(x, y) = \bar{f}_i(x) + \tilde{f}(x, y)$$

for some functions $\bar{f}_i(x)$ and $\tilde{f}(x, y)$.

If Assumption [**A1**] is satisfied and Assumption [**A2**] is not, then we can solve reduced versions of (4.1), (4.5), (4.6), and (4.7) with common multipliers (λ, μ, σ) but cannot make further reductions. When Assumptions [**A1**] and [**A2**] are satisfied, we are left with the reduced price-consistent multi-leader-common-follower game:

$$
\begin{array}{lll}
x_i^* & \in & \left\{ \begin{array}{ll} \underset{x_i \geq 0, \bar{g}_i(x_i) \geq 0}{\operatorname{argmin}} & \bar{f}_i(\hat{x}_i) + \tilde{f}(\hat{x}_i, y^*) - (\lambda^*)^T \tilde{g}(\hat{x}_i, y^*) - (\mu^*)^T (h(\hat{x}_i, y^*) - s^*) \end{array} \right\} \\
 & & \forall i = 1, \ldots, k \\
(y^*, s^*) & \in & \left\{ \begin{array}{ll} \underset{y \geq 0, s \geq 0}{\operatorname{argmin}} & \tilde{f}(x^*, y) - (\lambda^*)^T \tilde{g}(x^*, y) - (\mu^*)^T (h(x^*, y) - s) + \sigma^* s^T y \end{array} \right\} \\
(\lambda^*, \mu^*, \sigma^*) & \in & \left\{ \begin{array}{ll} \underset{\lambda \geq 0, \mu, \sigma \geq 0}{\operatorname{argmin}} & \lambda^T \tilde{g}(x^*, y^*) + \mu^T (h(x^*, y^*) - s^*) - \sigma (s^*)^T y^* \end{array} \right\},
\end{array}
\tag{5.4}
$$

where y and s need be optimal for only a single optimization problem. The stationarity conditions for this Nash game form a square nonlinear complementarity problem. Applying the inverse operation to σ leads to the equivalent formulation:

$$\begin{aligned} x_i^* &\in \left\{ \underset{x_i\geq 0, \bar{g}_i(x_i)\geq 0}{\operatorname{argmin}} \quad \bar{f}_i(\hat{x}_i) + \tilde{f}(\hat{x}_i, y^*) - (\lambda^*)^T \tilde{g}(\hat{x}_i, y^*) - (\mu^*)^T (h(\hat{x}_i, y^*) - s^*) \right\} \quad \forall i = 1, \ldots, k \\ (y^*, s^*) &\in \left\{ \begin{array}{ll} \underset{y,s}{\operatorname{argmin}} & \tilde{f}(x^*, y) - (\lambda^*)^T \tilde{g}(x^*, y) - (\mu^*)^T (h(x^*, y) - s) \\ \text{subject to} & 0 \leq y \perp s \geq 0 \end{array} \right\} \\ (\lambda^*, \mu^*) &\in \left\{ \underset{\lambda\geq 0, \mu}{\operatorname{argmin}} \quad \lambda^T \tilde{g}(x^*, y^*) + \mu^T (h(x^*, y^*) - s^*) \right\}, \end{aligned} \tag{5.5}$$

where σ is the multiplier on the complementarity constraint when writing the strong stationarity conditions. This development motivates the following definition.

Definition 5.1 *The game (5.5) is called a* price-consistent *multi-leader-common-follower game.*

The NCP formulation of the price-consistent multi-leader-common-follower game (5.5) is defined as follows.

$$\begin{aligned} \nabla_{x_i} \bar{f}_i(x) + \nabla_{x_i} \tilde{f}(x, y) - \nabla_{x_i} \bar{g}_i(x_i) \rho_i - \nabla_{x_i} \tilde{g}(x, y) \lambda - \nabla_{x_i} h(x, y) \mu - \chi_i = 0 & \qquad \forall i = 1, \ldots, k \\ \nabla_y \tilde{f}(x, y) - \nabla_y \tilde{g}(x, y) \lambda - \nabla_y h(x, y) \mu - \psi + S\xi = 0 & \\ \mu - \sigma + Y\xi = 0 & \\ 0 \leq \bar{g}_i(x_i) \perp \rho_i \geq 0 & \qquad \forall i = 1, \ldots, k \\ 0 \leq \tilde{g}(x, y) \perp \lambda \geq 0 & \\ h(x, y) - s = 0 & \\ 0 \leq x_i \perp \chi_i \geq 0 & \qquad \forall i = 1, \ldots, k \\ 0 \leq y \perp \psi \geq 0 & \\ 0 \leq s \perp \sigma \geq 0 & \\ 0 \leq -Ys \perp \xi \geq 0 & \end{aligned} \tag{5.6}$$

Other versions of this NCP can be posed by replacing the complementarity conditions with (4.2), (4.3), or (4.4). For example, the bounded multiplier version would be

$$\begin{aligned}
\nabla_{x_i}\bar{f}_i(x) + \nabla_{x_i}\tilde{f}(x,y) - \nabla_{x_i}\bar{g}_i(x_i)\rho_i + \nabla_{x_i}\tilde{g}(x,y)\lambda_i - \nabla_{x_i}h(x,y)\mu - \chi_i &= 0 && \forall i=1,\ldots,k\\
\nabla_y\tilde{f}(x,y) - \nabla_y\tilde{g}(x,y)\lambda_i - \nabla_y h(x,y)\mu - \psi + S\xi &= 0\\
\mu - \sigma + Y\xi &= 0\\
0 \le \bar{g}_i(x_i) \;\perp\; \rho_i &\ge 0 && \forall i=1,\ldots,k\\
0 \le \tilde{g}(x,y) \;\perp\; \lambda &\ge 0\\
h(x,y) - s &= 0\\
0 \le x_i \;\perp\; \chi_i &\ge 0 && \forall i=1,\ldots,k\\
0 \le \psi + s \;\perp\; y &\ge 0\\
0 \le \sigma + y \;\perp\; s &\ge 0\\
0 \le \psi + \sigma \;\perp\; \xi &\ge 0.
\end{aligned} \tag{5.7}$$

Both (5.6) and (5.7) are square NCPs without side variables and constraints, while the formulations using (4.2) and (4.3) for the complementarity constraints would have side variables. Moreover, χ_i can be eliminated from both (5.6) and (5.7) to produce further reduced models, and ψ and σ can be eliminated from (5.6).

By making a further simplifying assumption on the underlying model, we can establish an interesting relationship between price consistency and a multiobjective optimization problem. We start by defining *complete separability*.

Definition 5.2 *We say that the multi-leader-follower game (3.1) is* completely separable *if the general constraints consist of a set of constraints independent of other decision variables and a set of constraints common across all players, that is,*

$$g_i(x,y) = \begin{bmatrix} \bar{g}_i(x_i) \\ \tilde{g}(x,y) \end{bmatrix},$$

and the objective function consists of a separable term and a term common across all leaders, that is,

$$f_i(x,y) = \bar{f}_i(x_i) + \tilde{f}(x,y),$$

for all $i = 1,\ldots,k$.

Note that this definition differs from **[A1]** and **[A2]** in that the objective function must now be separable in x_i and y for all $i = 1,\ldots,k$. The following proposition relates completely separable, price-consistent multi-leader-common-follower games to a standard MPEC.

Proposition 5.1 *Assume that the multi-leader-common-follower game is completely separable. Then it follows that the first-order conditions (5.6) of the game (5.5) are equivalent to the strong stationarity conditions of the following MPEC:*

$$\begin{array}{lll} \underset{x\geq 0,y,s}{\text{minimize}} & \sum_{i=1}^{k} \bar{f}_i(x_i) + \tilde{f}(x,y) & \\ \text{subject to} & \bar{g}_i(x_i) \geq 0 & \forall i = 1,\ldots,k \\ & \tilde{g}(x,y) \geq 0 & \\ & h(x,y) - s = 0 & \\ & 0 \leq y \ \perp \ s \geq 0. & \end{array} \tag{5.8}$$

Proof. The proof follows by comparing the strong-stationarity conditions of the MPEC (5.8) with the first-order conditions of (5.6). □

Problem (5.8) minimizes the collective losses for the leaders and can be interpreted as finding a particular solution to a multiobjective optimization problem by minimizing a convex combination of all leaders' objectives. As a consequence, existence results can be derived for completely separable EPECs by showing the existence of a solution to the price-restricted MPEC (5.8). This observation provides a starting point for deriving existence results for certain classes of EPECs.

If there are no followers, that is, we are really solving a Nash game, problem (5.8) reverts to a standard nonlinear programming problem. Some traffic equilibrium models, for example, use this equivalence to reduce the problem of computing equilibrium traffic patterns to solving a single multicommodity network flow problem [9].

By introducing the price-consistent restriction, we produce a model that may be easier to solve than the original. Because price consistency is a restriction, any solution to the restricted model is a solution to the unrestricted version. However, the restricted model may not have a solution, while the unrestricted model may have a solution. Therefore, one would attempt to first solve the price-consistent multi-leader-common-follower game if possible and then resort to one of the general NCP or NLP reformulations from Section 4 if no solution is found.

The following example of a generalized Nash game shows the different possible results for the price-consistent formulation:

$$\begin{array}{lll} \underset{x_1}{\text{minimize}} & x_1^2 + a x_1 x_2 & \text{subject to } x_1 + x_2 = c \\ \underset{x_2}{\text{minimize}} & x_2^2 + b x_1 x_2 & \text{subject to } x_1 + x_2 = c, \end{array}$$

where a, b, and c are parameters. One can show that every point $(x_1, c - x_1)$ is an equilibrium point of the multi-leader-common-follower game (3.1). However, the price-consistent game can have zero, one, or an infinite number of solutions. In particular, the price-consistent game has the following:

1. A unique equilibrium if $a + b \neq 4$.
2. An infinite number of equilibria $(x_1, c - x_1, 2c)$ when $a = b = 2$.
3. An infinite number of equilibria $(x_1, c - x_1, 2x_1 - ax_1)$ when $a + b = 4$, $a \neq b$, and $c = 0$.
4. No equilibrium when $a + b = 4$, $a \neq b$, and $c \neq 0$.

In the last case, one player would make an infinite profit, while the other an infinite loss as x_1 goes to infinity and x_2 goes to minus infinity. Moreover, if $a = b$, the price-consistent game is computing a first-order critical point for the single optimization problem:

$$\underset{x_1, x_2}{\text{minimize}} \quad x_1^2 + x_2^2 + ax_1x_2 \quad \text{subject to } x_1 + x_2 = c.$$

This problem is unbounded below but has the unique critical point $(\frac{c}{2}, \frac{c}{2}, c + \frac{ca}{2})$ corresponding to a maximizer when $a > 2$, an infinite number of solutions $(x_1, c - x_1, 2c)$ when $a = 2$, and a unique solution $(\frac{c}{2}, \frac{c}{2}, c + \frac{ca}{2})$ when $a < 2$.

6 Numerical Experience

This section provides numerical experience in solving medium-scale EPECs with up to a few hundred variables. The numerical solution of EPECs is a novel area; there are no established test problem libraries and few numerical studies. We begin by describing the test problems and then provide a detailed comparison of our formulations with the diagonalization approach and the approach of [32]. All problems are available online at `http://www-unix.mcs.anl.gov/~leyffer/MacEPEC/`.

6.1 Description of Test Problems

The test problems fall into three broad classes: randomly generated problems, academic test problems, and a more realistic model that arose from a case study of the interaction of electric power and NO_x allowance markets in the eastern United States [4].

The `AMPL` models of all test problems identify the NCPs (4.1) and (4.5), the MPEC (4.6), and the NLP (4.7) formulations as `*-NCP.mod, *-NCPa.mod, *-MPEC.mod`, and `*-NLP.mod`, respectively. The price-consistent models (5.6) and (5.7) are labeled `*-PC.mod` and `*-PCa.mod`. The diagonalization techniques are also implemented in `AMPL`. The Gauss-Jacobi iteration is identified by `*-GJ.mod` and the Gauss-Seidel iteration by `*-GS.mod`. The NCP smoothing technique of [32] is identified by the addition of `*-NCPt.mod`.

6.1.1 Randomly Generated EPECs

Randomly generated test problems are usually a poor substitute for numerical experiments. However, the fact that solving EPECs is a relatively new area means there are few realistic test problems. Thus, in order to demonstrate the efficiency of our approach on medium-sized problems, we decided to include results on randomly generated problems.

We have written a random EPEC generator in `matlab` that generates a random EPEC instance and writes the data to an `AMPL` file. Each leader is a quadratic program with equilibrium constraints (QPEC) and follows ideas from [12]. We note that [32] has a more sophisticated generator that follows ideas from [14]. Each leader $i = 1, \ldots, k$ has variables $x_i \in R^n$, where we assume for simplicity that all leaders have the same number of variables. Leader i's problem is the QPEC

$$\begin{array}{ll} \underset{x_i \geq 0, y, s}{\text{minimize}} & \frac{1}{2} x^T G_i x + g_i^T x \\ \text{subject to} & b_i - A_i x_i \geq 0 \\ & Nx + My + q = s \\ & 0 \leq y \ \perp \ s \geq 0, \end{array}$$

where the data for leader i is given by the following randomly generated vectors and matrices: g_i, the objective gradient; G_i, the objective Hessian, a positive definite $nk \times nk$ matrix; A_i, the $p \times n$ constraint matrix on the controls x_i; and b_i, a random vector for which $b_i - A_i e \geq 0$. The data for the follower is given by N, an $m \times nk$ matrix; M, an $m \times m$ diagonally dominant matrix; and the vector q. These problems satisfy Assumptions [**A1**] and [**A2**], so a price-consistent version can be generated, but the problem is not completely separable.

The data is generated randomly from a uniform $(0, 1)$ distribution and is scaled and shifted to lie in a user-defined interval. The generated problems are sparse so that large-scale EPECs can be generated and solved. The `AMPL` model files are `EPEC-*.mod`. We have generated three datasets, each containing ten instances. The characteristics of each dataset are shown in Table 1. Note that the data is deliberately output in single precision because, in our experience, this heuristic usually increases degeneracy.

6.1.2 Academic Test Problems

ex-001 is a small EPEC having an equilibrium point. Assumptions [**A1**] and [**A2**] are satisfied by this problem, so price-consistent formulations exist. The `AMPL` models are `ex-001-*.mod`.

example-4 is a small EPEC from [27], similar to **ex-001** but designed to illustrate a situation where each Stackelberg game has a solution but no solution exists for the multi-leader-common-follower game. In particular, this problem is infeasible. Moreover, it

Table 1: Details of Randomly Generated Datasets

	Datasets		
Parameter	01–10	11–20	21–30
Number of leaders, l	2	2	4
Number of leader constraints, p	4	8	8
Number of follower variables, m	16	32	32
Number of leader variables, n	8	16	16
Coefficient range of A_i	$[-4, 4]$	$[-4, 4]$	$[-4, 4]$
Coefficient range of N	$[0, 8]$	$[0, 8]$	$[0, 8]$
Coefficient range of b	$[0, 8]$	$[0, 8]$	$[0, 8]$
Coefficient range of g_i	$[-6, 6]$	$[-6, 6]$	$[-6, 6]$
Coefficient range of q	$[-4, 4]$	$[-4, 4]$	$[-4, 4]$
Density of G	0.2	0.2	0.05
Density of A_i	0.4	0.4	0.2
Density of N	0.2	0.2	0.1
Density of M	0.2	0.2	0.1

violates Assumption [**A2**], so there is no square price-consistent model. The `AMPL` models are `ex-4-*.mod`.

outrata3 is an EPEC generated from the MPEC models in [15, 26]. The control variables from all leaders enter the lower-level problem by averaging over the leaders. This trick ensures that the EPEC does not separate into individual MPECs. All Stackelberg players have the same constraints but different objective functions, and the problem violates Assumption [**A2**]. The `AMPL` models are `outrata3-*.mod`.

outrata4 is derived from **outrata3** so that the objective functions satisfy Assumption [**A2**]. A price-consistent solution to this model exists. The `AMPL` models are contained in `outrata4-*.mod`.

6.1.3 Electricity Market Models

This model is an electric power market example from [27]. It has two electricity firms (leaders) competing in a market, an arbitrager (follower) that exploits the price differential between regions, and an independent system operator (ISO). Unlike the formulation in [27], however, we enforce the response of the followers to be identical for all leaders.

Each leader maximizes profit subject to capacity constraints, the arbitrager's optimality conditions, and a market clearing condition. This optimization problem is an MPEC. The competition between leaders gives rise to a complementarity problem obtained by

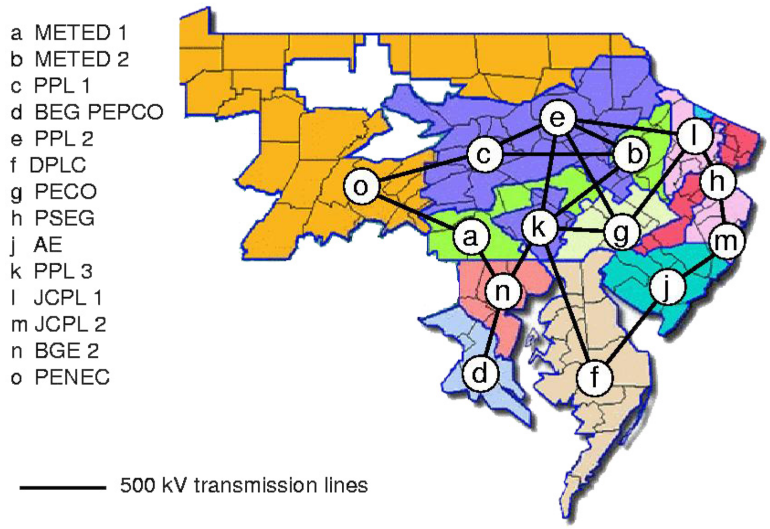

Figure 2: Network topology of `electric-3.dat`

writing down the strong stationarity conditions for each leader, the ISO's optimality conditions, and the market clearing condition.

The models are denoted by `electric-*.mod`. There are three data instances. The first, `electric-1.dat` is the small 3-node example from [27]. Problem `electric-2.dat` is another small 3-node example. The data for `electric-3.dat` is a larger 14-node example derived from real data of the PJM region [4]. The network of `electric-3.dat` is shown in Figure 2. The 14-node example results in a game with approximately 150 constraints and 160 variables, making this the largest EPEC solved to date. Price-consistent formulations do not exist for these problems because the objective functions of the leaders violate Assumption [**A2**].

6.2 Comparison of EPEC Solution Approaches

The different formulations give rise to problems of differing size, which are summarized in Tables 2 and 3. Each triple in these tables shows the number of variables, the number of constraints, and the number of complementarity conditions *after* `AMPL`*'s presolve*, which may eliminate variables and constraints. Models PC and PCa were run without `AMPL`'s presolve, because it can destroy the square structure. For the random problems, we set `option presolve_eps 1E-14` to avoid error messages from `AMPL`'s presolve.

The benefit of the diagonalization techniques is that they do not require the strong stationarity conditions to be written as `AMPL` models, a process that is potentially error prone because the gradients and Jacobians need to be computed by hand for each MPEC. In addition, the MPEC problems solved are much smaller than the entire EPEC. On the other hand, each sweep of a diagonalization technique solves k MPECs.

Table 2: Test Problem Sizes for Different Formulations

	Formulation			
Problem	NCP	NCPa	MPEC	NLP
ex-001	12/ 14/ 6	14/ 16/ 6	14/ 16/ 6	12/ 8/-
example-4	14/ 16/ 10	16/ 18/ 10	15/ 16/ 9	16/ 8/-
outrata3	81/105/ 56	81/ 105/ 56	80/ 92/ 44	84/ 52/-
outrata4	80/104/ 56	80/ 104/ 56	80/ 92/ 40	84/ 52/-
electric-1	153/180/110	228/ 255/110	165/168/ 86	225/117/-
electric-2	157/182/114	232/ 257/114	169/170/ 90	229/118/-
electric-3	734/806/560	1022/1094/560	806/806/488	990/502/-
random-[01-10]	193/199/ 76	204/ 231/ 97	193/199/ 76	208/120/-
random-[11-20]	375/379/137	393/ 443/183	375/379/137	416/240/-
random-[21-30]	567/494/112	592/ 642/235	567/494/112	750/472/-

We use `filterMPEC` [15, 10], a sequential quadratic programming (SQP) method adapted to solving MPECs to solve the `AMPL` models `*-MPEC.mod, *-NCP.mod, *-NCPa.mod` and `*-NLP.mod`. In addition, we use the NCP solver `PATH` [5, 8], a generalized Newton method that solves a linear complementarity problem to compute the direction, for the price-consistent models. We also experimented with using `PATH` to solve the other NCP formulations, but our experience with these nonsquare and degenerate NCPs was rather disappointing.

Table 3: Test Problem Sizes for Different Formulations

	Formulation			
Problem	NCP(t)	PC	PCa	GJ/GS
ex-001	12/ 14/ 6	8/ 8/ 3	8/ 8/ 3	3/ 2/ 1
example-4	14/ 16/ 10	n/a	n/a	3/ 2/ 1
outrata3	81/105/ 56	n/a	n/a	9/ 8/ 4
outrata4	80/104/ 52	28/ 28/ 16	28/ 28/ 16	8/ 7/ 4
electric-1	153/146/ 74	n/a	n/a	48/ 32/12
electric-2	157/148/ 78	n/a	n/a	48/ 32/12
electric-3	734/664/416	n/a	n/a	230/171/72
random-[01-10]	187/219/107	136/136/ 72	136/136/ 72	29/ 19/13
random-[11-20]	371/435/211	272/272/144	272/272/144	55/ 39/23
random-[21-30]	639/824/375	352/352/192	352/352/192	70/ 31/ 6

Table 4 provides a comparison of iteration counts for the different solution approaches for EPECs. For the NCP/MPEC/NLP formulations of Section 3 we report the number of major (SQP) iterations. For the price-consistent NCP formulations of Section 5 we count the number of `PATH`'s major iterations (roughly equivalent to an SQP iteration). For the sequential NCP approach we report the total number of major (SQP) iterations. Finally, for the diagonalization methods we sum the average number of major iterations to solve each MPEC during the Gauss-Jacobi or Gauss-Seidel process. We believe this average provides an accurate picture of the relative performance of the diagonalization methods, which solve smaller (single leader) subproblems. The iteration counts for the randomly generated EPECs are averaged over the ten problem instances. Comparing CPU times would have been problematic, because some of the algorithms are implemented as `AMPL` scripts, which run significantly slower than Fortran or C. The tolerance of Gauss-Seidel and Gauss-Jacobi were set to 10^{-6}, and the relative tolerance of the NLP solver is 10^{-6}.

The column headers in Table 4 refer to the problem name and the solution approach, where NCP, NCPa, MPEC, and NLP refer to formulations (4.1), (4.5), (4.6), and (4.7), respectively; PC and PCa refer to the price-consistent formulations (5.6) and (5.7); NCP(t) refers to the approach in [32] with the standard sequence of smoothing parameters $t = 1, 10^{-1}, \ldots, 10^{-6}$ (except that we may terminate early if the solution is complementary); and GJ and GS refer to the Gauss-Jacobi and Gauss-Seidel method, respectively.

Table 4: Comparison of Iteration Counts for EPEC Methods

	Solution Methods								
Problem	NCP	NCPa	MPEC	NLP	NCP(t)	GJ	GS	PC	PCa
ex-001	1	3	1	1	17	2.3	2	2	2
example-4	7[I]	16[I]	10[I]	35[S]	80[I]	19.6[S]	18.6[S]	n/a	n/a
outrata3	40[I]	41[I]	22[I]	34	69[I]	CYCLE	CYCLE	n/a	n/a
outrata4	14	24	84	11	51	13.25	2	8	7
electric-1	32[I]	65[I]	28[I]	116[I]	55[I]	CYCLE	3.5	n/a	n/a
electric-2	8	32	13	7	32	105.5	52.0	n/a	n/a
electric-3	48	47[I]	28	70[I]	36	1.0	1.0	n/a	n/a
random[01-10]	114.4	27.1	13.4	2.1	690.6	16.5	11.6	8.4	8.2
random[11-20]	38.8	47.6	19.0	6.0	1083.5	62.1	35.9	11.3	9.2
random[21-30]	20.8	34.8	17.1	3.8	192.5	47.2	30.0	10.1	8.3

In Table 4, we tag solvers that converge to an infeasible solution with [I]. Cycling in the Gauss-Jacobi method is indicated by CYCLE. Finally, solution to spurious stationary points that are not strongly stationary is indicated by [S]. We note, that for `example-4-*`, the NCP/MPEC approaches successfully detect infeasibility, while NLP and the diagonalization techniques converge to spurious stationary points where one or both players have trivial descent directions. Thus, even though Gauss-Seidel and Gauss-Jacobi converge, the result is misleading because it does not correspond to a solution. This result is not surprising because the underlying MPEC solver is not guaranteed to converge to strongly stationary points. Currently, no MPEC solver guarantees strong stationarity under reasonable assumptions.

On the other hand, Gauss-Seidel is the only solver that finds a feasible point for `electric-1`, while all other solvers fail to find a feasible point. In our view, these examples illustrate the need for alternative models and solution methodologies to solve these challenging optimization problems.

6.2.1 Comparing the Performance of the Solvers

Regarding the performance of the different solvers and formulations, there appears to be no clear overall winner, though the NLP approach is often the fastest. In particular, for the randomly generated problems, the NLP solver can be orders of magnitude faster than the other approaches.

We also note that the price-consistent formulations are very competitive when they exist and have a solution. This result indicates that more research is needed to identify robust formulations and solution tools for EPECs. A related open question concerns the "correct" formulation and presolve for NCPs. A bad formulation can often hide structure such as skew-symmetry that `PATH` can exploit during the solution.

When we compare the direct NCP approach (4.5) with the sequential NCP(t) approach, we observe that there is no benefit in smoothing the NCP formulation. The direct approach is typically an order of magnitude faster than the sequential approach. Contrary to intuition, the sequential NCP approach does not benefit from warm starts (with the exception of `electric-3`): each NCP takes a similar number of iterations as t is reduced. This observation is consistent with the situation in MPECs, where SQP methods are much faster than sequential relaxation approaches that solve one NLP per iteration for a decreasing sequence of regularization parameters. In our view, relaxation approaches are best used in conjunction with inexact solves, such as in the context of interior-point methods [16, 28].

6.2.2 Comparing the Solutions Obtained by the Solvers

We have also compared the primal solutions for each of the leaders in an effort to determine whether the different solvers converge to a similar solution. For the first two examples, `ex-001` and `example-4`, all solvers (except for the diagonalization approaches and NLP) obtain the same solution. However, for some of the remaining problems, the nonconvexity of EPECs produces different solutions.

outrata3: only NLP converges to a strongly stationary point. NCP and NCP(t) obtain nearly feasible solutions (to within 10^{-5}). Gauss-Seidel and Gauss-Jacobi cycle without finding a stationary point.

outrata4: all solvers converge to strongly stationary points. However, all solutions differ. This problem is highly nonconvex, and the stationary points are at best local Nash equilibria.

electric-1: only Gauss-Seidel converges to a strongly stationary solution. Gauss-Jacobi enters a cycle that appears to be close to a stationary point. NLP is the only other solver that gets close (10^{-2}) to feasibility.

electric-2: all solvers converge to a stationary point. MPEC, NLP, GS, and GJ converge to a solution with the same revenue for the leaders. However, NCP(t) is able to find a stationary point that improves the revenue of both leaders.

electric-3: NCP, MPEC, NCP(t), GS, and GJ converge to the trivial solution (generation, wheeling, and sales are all zero), while NLP and NCPa fail to find a feasible point.

random Of the 30 random problems, only two produce the same solution for all solvers. In 29 instances, GS, GJ, PC, PCa, and NLP converge to the same solution, while the other solvers find alternative solutions.

Our experience demonstrates the value of different solution approaches. In particular, the NLP and the price-consistent approaches are valuable alternatives to the standard Gauss-Seidel/Jacobi iterations. Different approaches allow us to detect multiple solutions and infeasibility.

7 Conclusions

We have presented a characterization of the solution set to multi-leader-common-follower games or EPECs and two novel approaches for solving them. The first approach is based on the strong stationarity conditions of each leader, and we derive a family of NCP, NLP, and MPEC formulations. The second approach imposes an additional restriction, called price consistency, that results in a square nonlinear complementarity problem. Both approaches allow the use of standard nonlinear optimization software to be extended to EPECs. In both approaches, the EPEC is solved by a single optimization problem, unlike traditional approaches that solve a sequence of related optimization problems.

We provide numerical results demonstrating that our new approaches are competitive with existing methods in terms of both robustness and efficiency. The new NLP/NCP based approaches presented here provide a useful alternative to traditional diagonalization techniques. In particular, the NLP and the price-consistent formulations can be competitive with diagonalization techniques.

A number of important open research questions remain. For example, all solution techniques for EPEC rely on efficient and robust MPEC solvers. To our knowledge, however, no MPEC solver can guarantee convergence to strongly stationary points under reasonable conditions. Another open question is how to formulate and presolve NCPs, MPECs, and EPECs so that the solvers can take advantage of underlying structure such as skew symmetry.

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