Stochastic Separable Mixed-Integer Nonlinear Programming via Nonconvex Generalized Benders Decomposition

Xiang Li

Process Systems Engineering Laboratory
Department of Chemical Engineering
Massachusetts Institute of Technology
xiangli@mit.edu
xiang.li@chee.queensu.ca

PhD Winter School 2011
Oppdal, Norway
March 10, 2011

Agenda

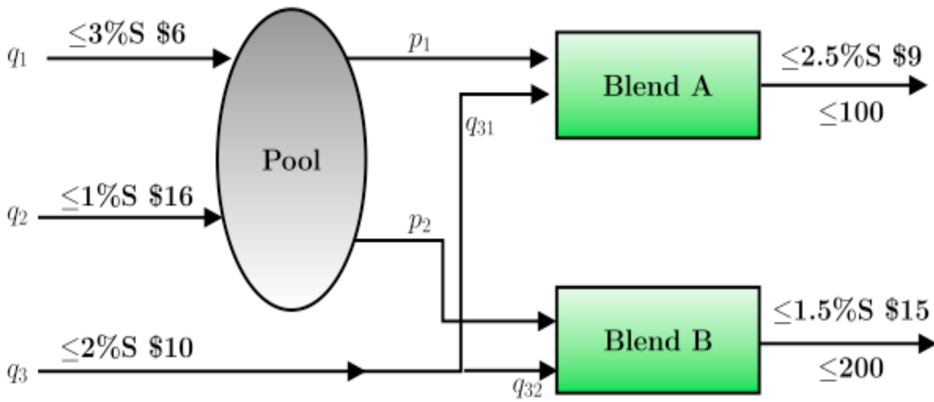
- Motivation
- Duality Theory – A Geometric Perspective
- Generalized Benders Decomposition
- Nonconvex Generalized Benders Decomposition
- Computational Study
- Summary

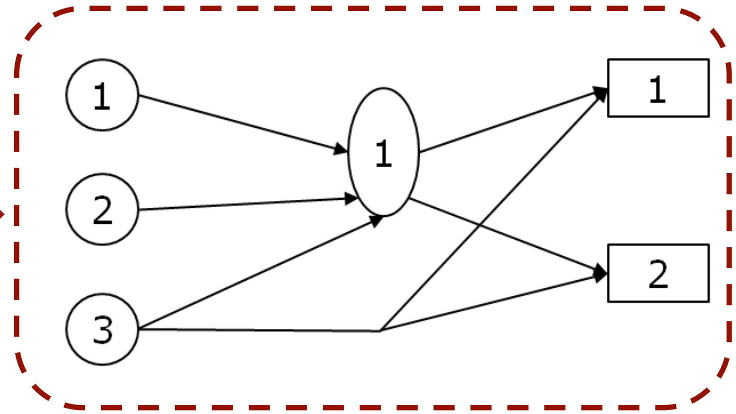
Agenda

- Motivation
- Duality Theory – A Geometric Perspective
- Generalized Benders Decomposition
- Nonconvex Generalized Benders Decomposition
- Computational Study
- Summary

Motivation

Haverly pooling problem





Haverly pooling problem *

$$\max \quad 9(p_1 + q_{31}) + 15(p_2 + q_{32}) - (6q_1 + 16q_2 + 10q_3)$$

s.t.

$$q_1 + q_2 = p_1 + p_2$$

$$3q_1 + q_2 = x_s(p_1 + p_2)$$

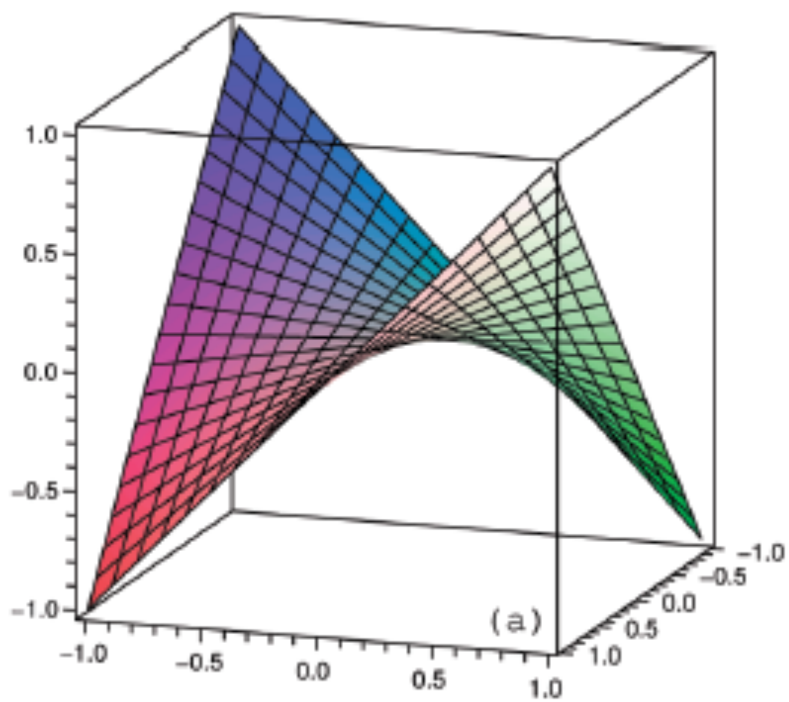
$$q_{31} + q_{32} = q_3$$

Linear inequalities

* Haverly C. A., "Studies of the behaviour of recursion for the pooling problem", *ACM SIGMAP Bulletin, 25:29-32, 1978.*

Motivation

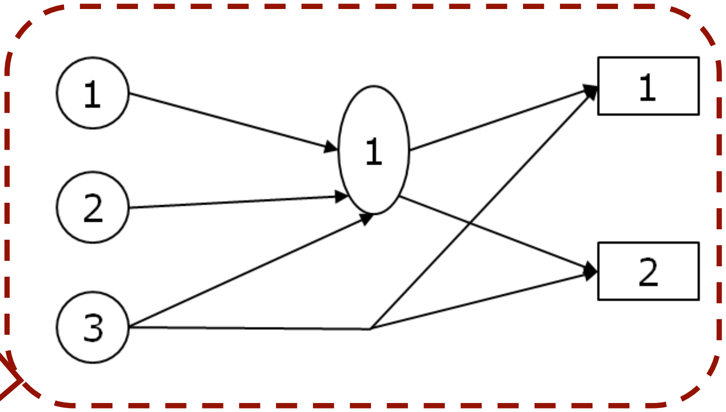
Nonconvexity of bilinear function

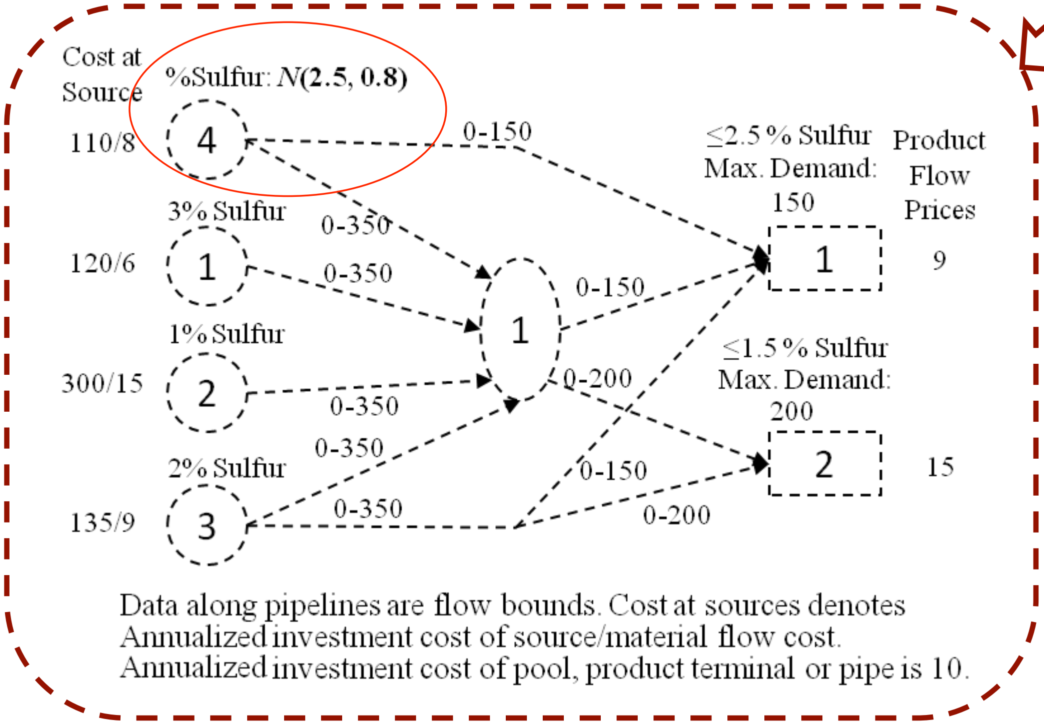


Bilinear function

Motivation

Stochastic Haverly pooling problem





Motivation

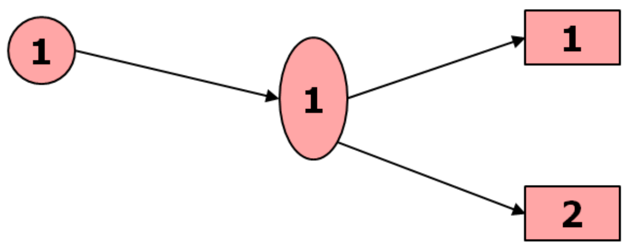
Stochastic Haverly pooling problem – Two-stage stochastic program

The stochastic pooling problem
- A potentially large-scale mixed-integer bilinear program

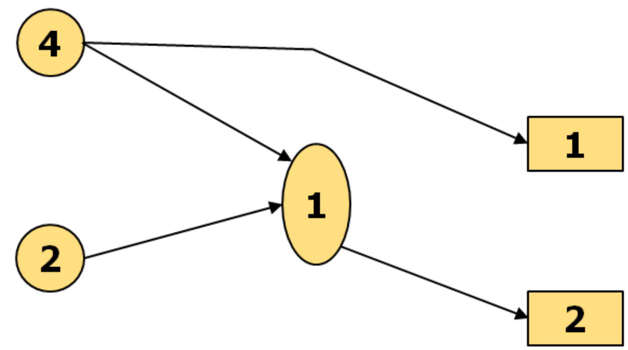
$$
P = \begin{cases}
\min\limits_{\substack{y, x_1, \dots, x_s, \\ q_1, \dots q_s, \\ u_1, \dots u_s}} \quad c_1^T y + \sum_{h=1}^{s} \left(c_{2,h}^T x_h + c_{3,h}^T q_h + c_{4,h}^T u_h \right) & \text{Economic objective} \\
s.t. \quad u_{h,l,t} = x_{h,l} q_{h,t}, \quad \forall (l,t) \in \Omega \subset \{1, \dots, n_x\} \times \{1, \dots, n_q\}, \ \forall h \in \{1, \dots, s\} & \text{Mass balances} \\
\tilde{A}_{2,h}^{(equ)} x_h + \tilde{A}_{3,h}^{(equ)} q_h + \tilde{A}_{4,h}^{(equ)} u_h = \tilde{b}_h^{(equ)}, \ \forall h \in \{1, \dots, s\} & \\
A_{1,h} y + A_{2,h} x_h + A_{3,h} q_h + A_{4,h} u_h \le b_h, \ \forall h \in \{1, \dots, s\} & \text{Flow constraints} \\
\tilde{A}_{2,h} x_h + \tilde{A}_{3,h} q_h + \tilde{A}_{4,h} u_h \le \tilde{b}_h, \ \forall h \in \{1, \dots, s\} & \\
\mathrm{x}_h^L \le \mathrm{x}_h \le \mathrm{x}_h^U, \ q_h^L \le q_h \le q_h^U, \ \forall h \in \{1, \dots, s\} & \\
\mathrm{B} y \le d, & \text{Topology constraints} \\
x_h \in \mathbb{R}^{n_x}, q_h \in \mathbb{R}^{n_q}, u_h \in \mathbb{R}^{n_u}, \forall h \in \{1, \dots, s\}, \ y \in \{0,1\}^{n_y} &
\end{cases}
$$

Motivation

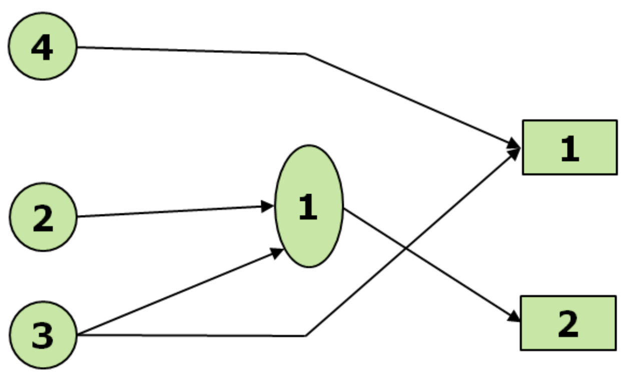
Stochastic Haverly pooling problem – Different Results



Form. 1 – Uncertainty and quality not addressed



Form. 2 – Only quality addressed

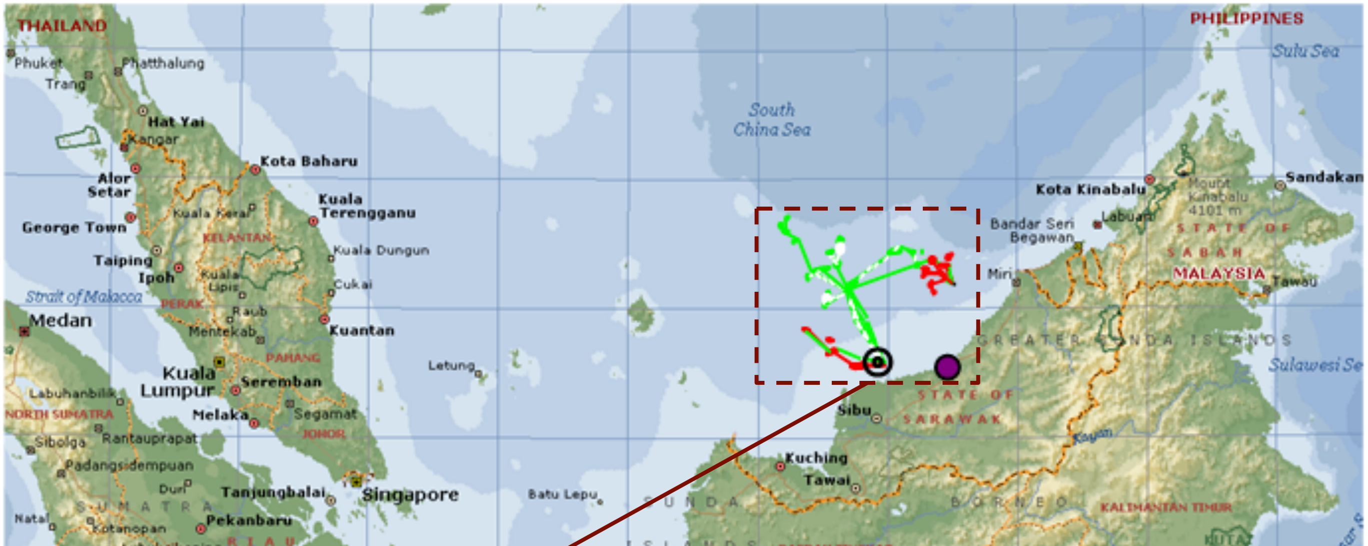


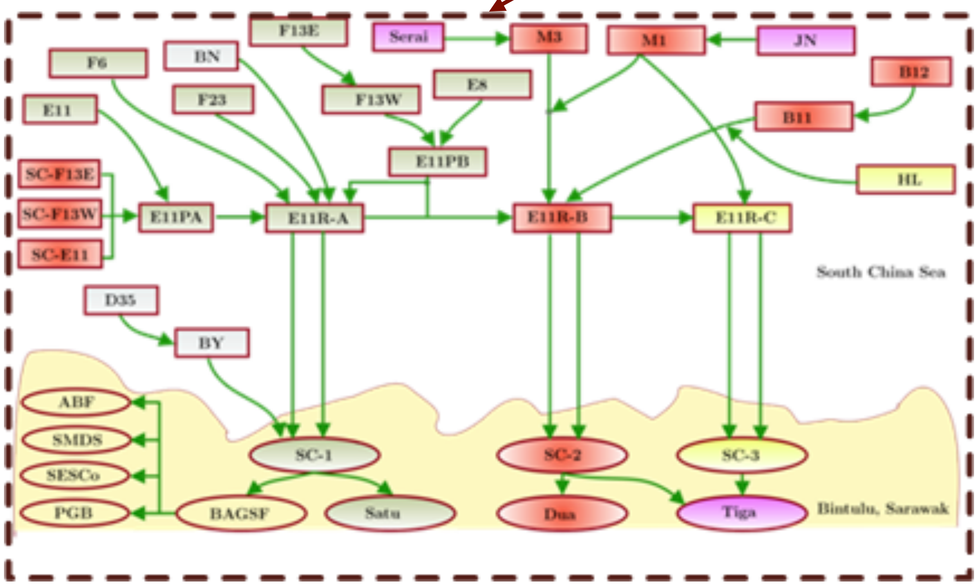
Form. 3 – Both uncertainty and quality addressed

Form.	*E*(profit)	Sulfur Spec. Satisfied?
1	N/A	No
2	104	Yes
3	118	Yes

Motivation

Sarawak gas production system (SGPS)

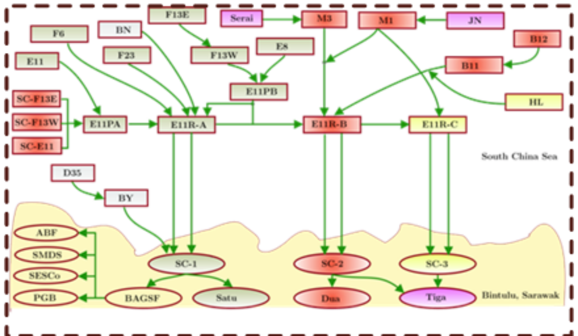


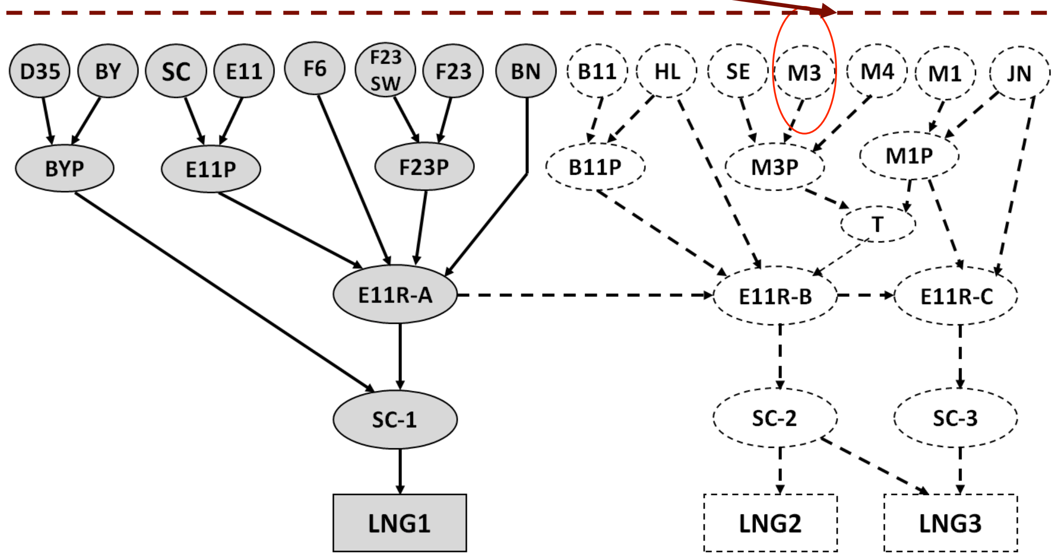


- Daily production
  - 4 billion scf
- Annual revenue
  - US $5 billion (4% of Malaysia's GDP)

Motivation

SGPS design problem





Gray symbols with solid lines:

- Units developed already.

Symbols with dashed lines:

- Units to be determined.

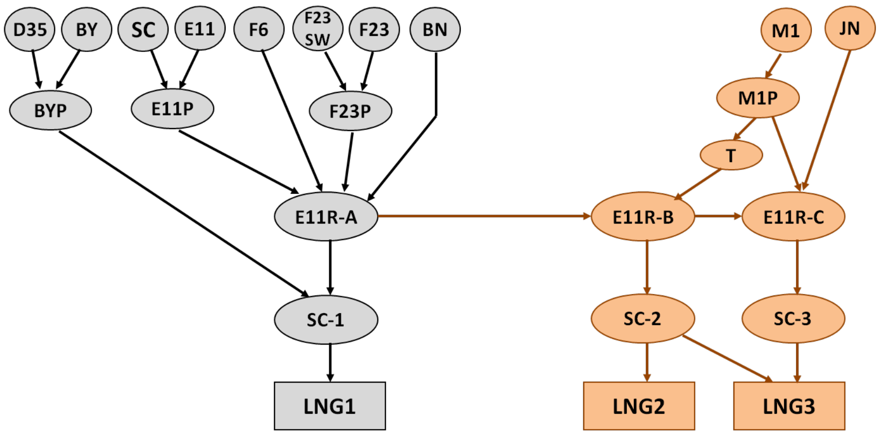
Source of uncertainty:

- CO_2 % of the gas from field M3.

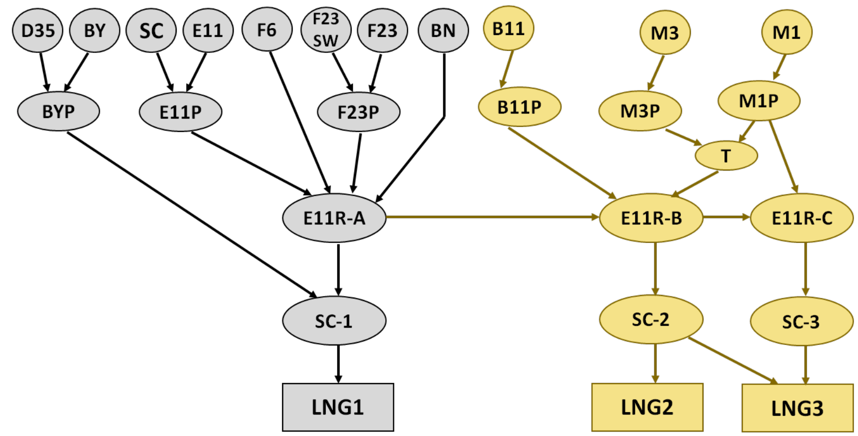
The design problem can be modeled as the stochastic pooling problem

Motivation

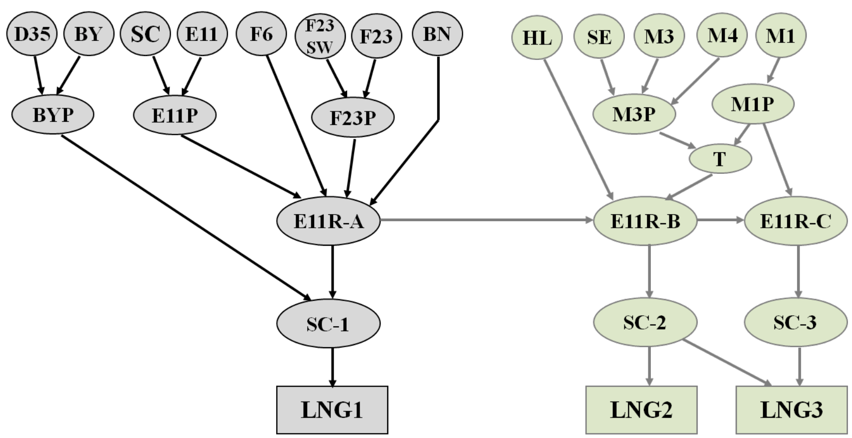
SGPS design problem – Different results



Form. 1 – Uncertainty and quality not addressed



Form. 2 – Only quality addressed



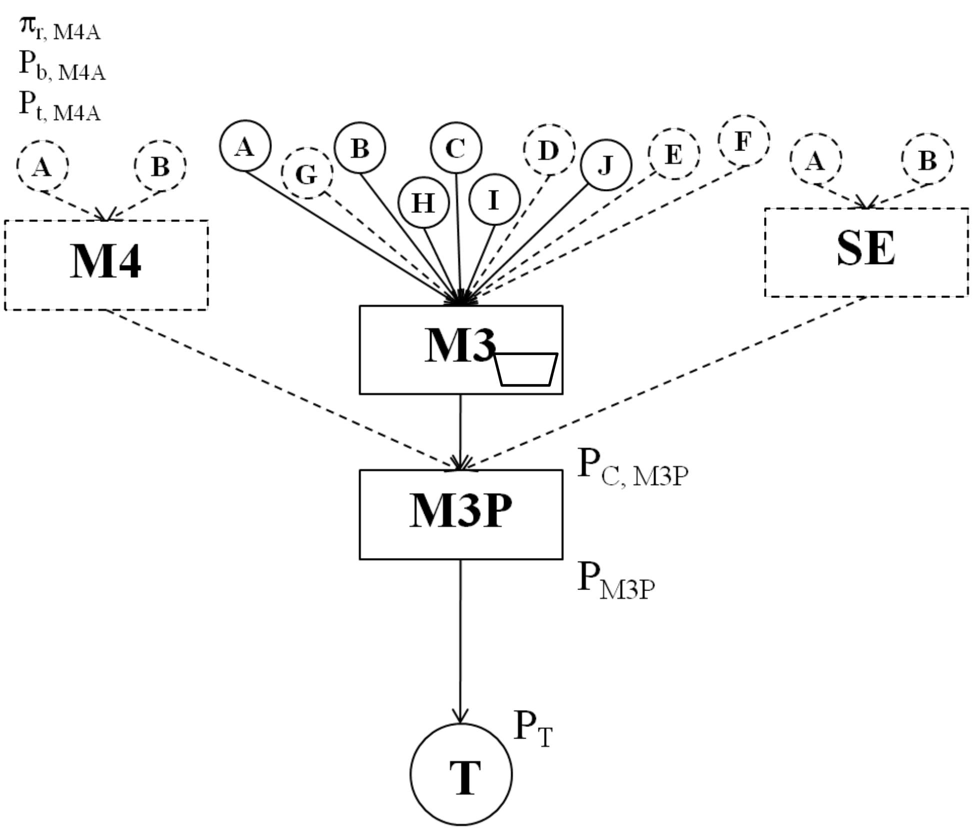
Form. 3 – Both uncertainty and quality addressed

Form.	*E*(NPV) (Billion $)	CO_2 Spec. Satisfied?
1	N/A	No
2	29.0	Yes
3	32.2	Yes

Motivation

SGPS design problem – More sophisticated model

A SGPS Subsystem



Well performance model (M4A)

$$\alpha_{M4A} F_{M4A,M4} + \beta_{M4A} F^2_{M4A,M4} = \pi^2_{r,M4A} - P^2_{b,M4A}$$

$$\vartheta_{M4A} F^2_{M4A,M4} = P^2_{b,M4A} - \lambda_{M4A} P^2_{t,M4A}$$

Compression model

$$W_{M3P} = \omega_{M3P} F_{M3P,T} \left[\left(\frac{P_{M3P}}{P_{c,M3P}} \right)^{\nu} - 1 \right]$$

Trunkline flow-pressure relationship

$$P^2_{M3P} - P^2_T = \kappa_{M3P,T} F^2_{M3P,T}$$

Motivation

Stochastic separable mixed-integer nonlinear programs

Stochastic separable mixed-integer nonlinear programs
– Models for energy infrastructure design and more

Natural Gas Production System

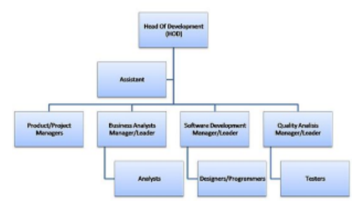
$$\min_{x_1,\ldots,x_s,y} \quad \sum_{h=1}^{s} w_h \left(f_h(x_h) + c_h^T y \right)$$

$$\begin{aligned} s.t. \quad & g_h(x_h) + B_h y \leq 0, \quad h = 1,\ldots,s \\ & x_h \in X_h, \quad h = 1,\ldots,s \\ & y \in Y \end{aligned}$$

Energy Polygeneration Plant

Pump Network

Oil Refinery



Software System

Motivation

Challenges for numerical optimization

Stochastic separable mixed-integer nonlinear programs
– Models for energy infrastructure design and more

$$\min_{x_1,\ldots,x_s,y} \sum_{h=1}^{s} w_h \left(f_h(x_h) + c_h^T y \right)$$

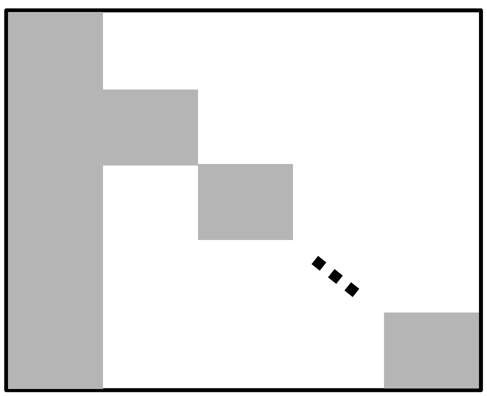
$$\begin{aligned} s.t. \quad & g_h(x_h) + B_h y \leq 0, \quad h = 1,\ldots,s \\ & x_h \in X_h, \quad h = 1,\ldots,s \\ & y \in Y \end{aligned}$$

- Accuracy of uncertainty representation
  - Large-scale optimization problems
- Profitability and feasibility of design and operation
  - Global optimization of nonconvex problems

Motivation

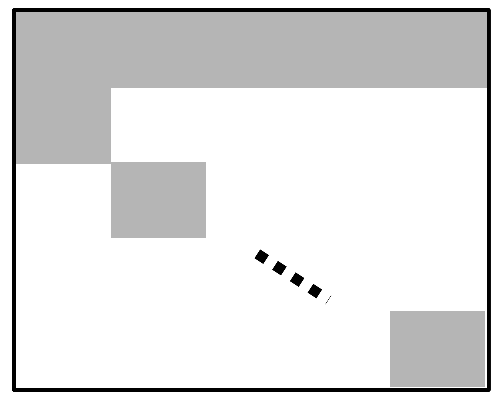
Two decomposition philosophies for scenario-based problems

$$
\begin{aligned}
\min_{x_1,\ldots,x_s,y} \quad & \sum_{h=1}^{s} w_h \left(f_h(x_h) + c_h^T y \right) \\
s.t. \quad & g_h(x_h) + B_h y \le 0, \quad h = 1,\ldots,s \\
& x_h \in X_h, \quad h = 1,\ldots,s \\
& y \in Y
\end{aligned}
$$



- Benders decomposition/L-shaped method
  - Linear duality
- Generalized benders decomposition
  - Nonlinear duality

$$
\begin{aligned}
\min_{x_1,\ldots,x_s,y} \quad & \sum_{h=1}^{s} w_h \left(f_h(x_h) + c_h^T y_h \right) \\
s.t. \quad & g_h(x_h) + B_h y_h \le 0, \quad h = 1,\ldots,s \\
& y_h = y_{h+1}, \quad h = 1,\ldots,s-1 \\
& x_h \in X_h, \quad h = 1,\ldots,s \\
& y_h \in Y
\end{aligned}
$$



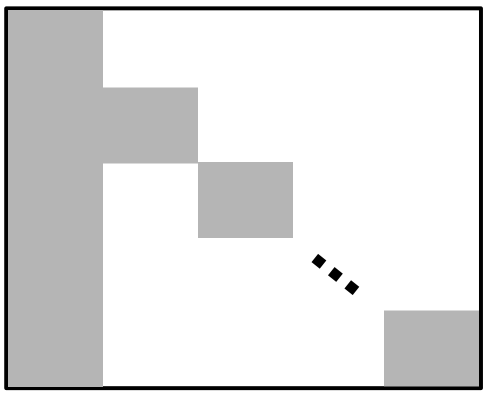
- Danzig-Wolfe decomposition
  - Linear duality
- Lagrangian relaxation
  - Nonlinear duality

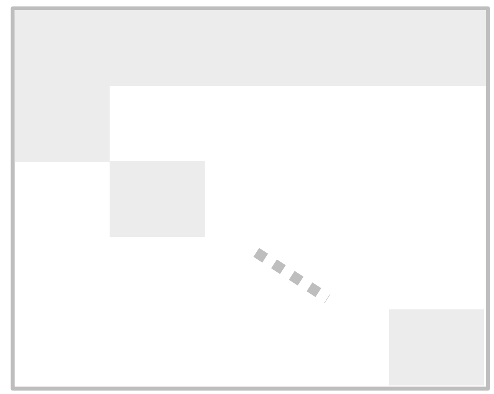
Motivation

Two decomposition philosophies for scenario-based problems

$$\min_{x_1,\dots,x_s,y} \sum_{h=1}^{s} w_h \left(f_h(x_h) + c_h^T y \right)$$
$$s.t. \quad g_h(x_h) + B_h y \le 0, \quad h = 1,\dots,s$$
$$x_h \in X_h, \quad h = 1,\dots,s$$
$$y \in Y$$

$$\min_{x_1,\dots,x_s,y} \sum_{h=1}^{s} w_h \left(f_h(x_h) + c_h^T y_h \right)$$
$$s.t. \quad g_h(x_h) + B_h y_h \le 0, \quad h = 1,\dots,s$$
$$y_h = y_{h+1}, \quad h = 1,\dots,s$$
$$x_h \in X_h, \quad h = 1,\dots,s$$
$$y_h \in Y$$



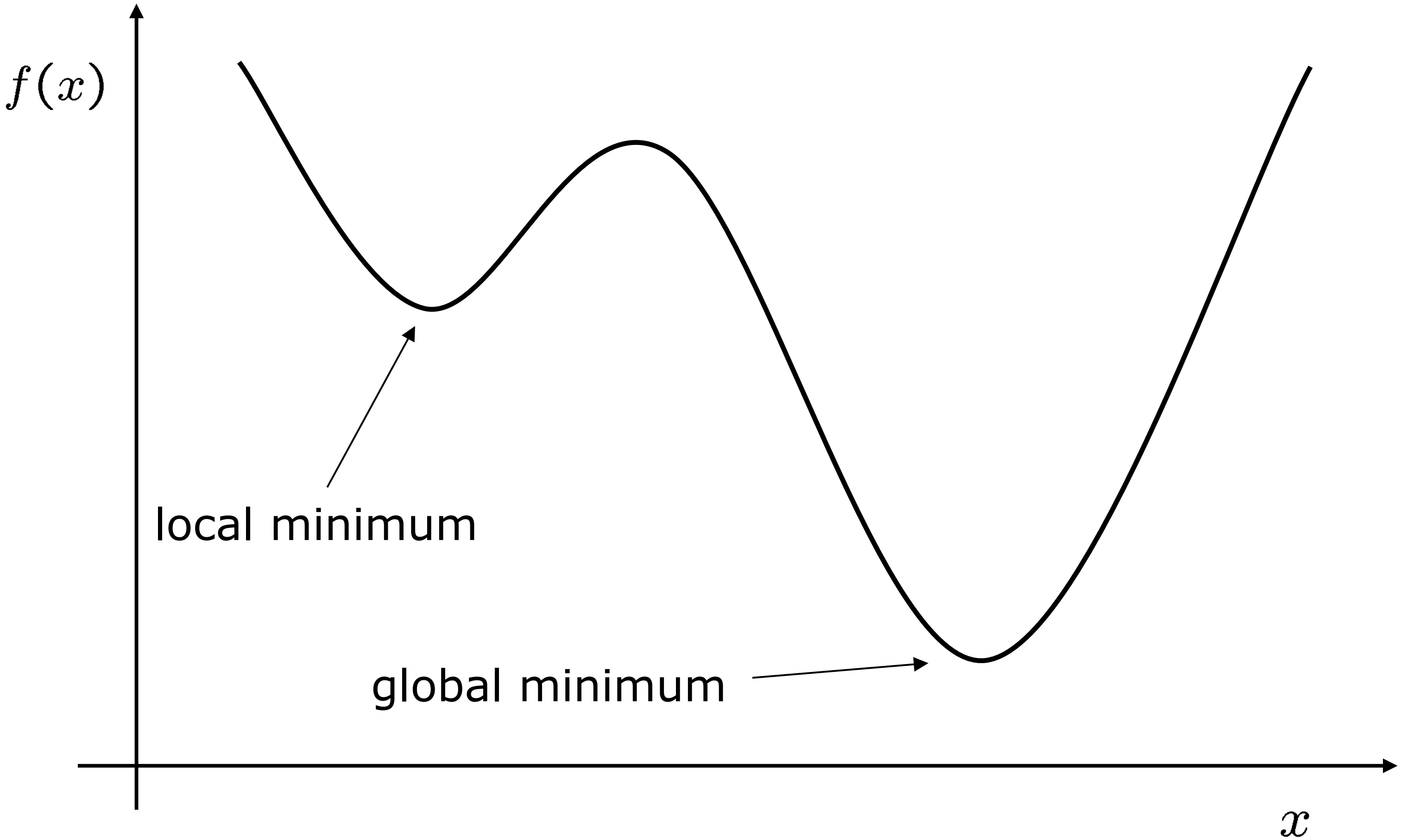


- Benders decomposition/L-shaped method
  - Linear duality
- Generalized Benders decomposition
  - Nonlinear duality

- Danzig-Wolfe decomposition
  - Linear duality
- Lagrangian relaxation
  - Nonlinear duality

Motivation

Nonconvex optimization



Standard optimization techniques cannot distinguish between suboptimal local minima

Motivation

Deterministic global optimization methods for nonconvex programs

- Deterministic global optimization via branch-and-bound

  - Generates a sequence of upper and lower bounds that converge to a global optimum by domain partitioning and restriction/relaxation.

- References

  - M. Tawarmalani and N. Sahinidis. *Convexification and global optimization in continuous and mixed-integer nonlinear programming*. Kluwer Academic Publishers, 2002.

  - R. Horst, P. M. Pardalos, and N. V. Thoai. *Introduction to Global Optimization*. Kluwer Academic Publishers, 2nd edition, 2000.

  - Floudas, C. A. Deterministic Global Optimization: Theory, Methods and Applications; Kluwer Academic Publishers, 2000.

- Commercial software

  - BARON (Branch-and-Reduce Optimization Navigator): Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010.

Agenda

- Motivation
- Duality Theory – A Geometric Perspective
- Generalized Benders Decomposition
- Nonconvex Generalized Benders Decomposition
- Compuational Study
- Summary

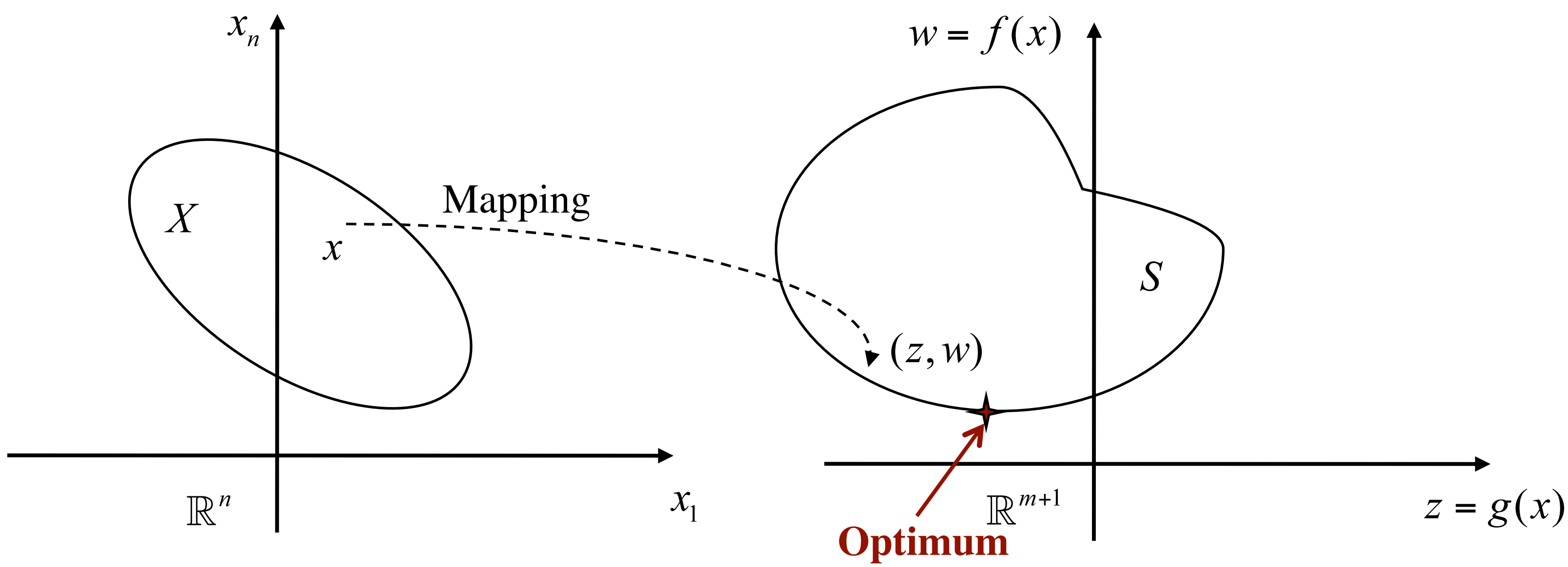
Duality Theory – A Geometric Perspective

Mapping into constraint-objective space

Primal Problem

$$\min_{x \in X} \; f(x)$$

$$s.t. \; g(x) \le 0 \qquad f : \mathbb{R}^n \to \mathbb{R}, \quad g : \mathbb{R}^n \to \mathbb{R}^m$$

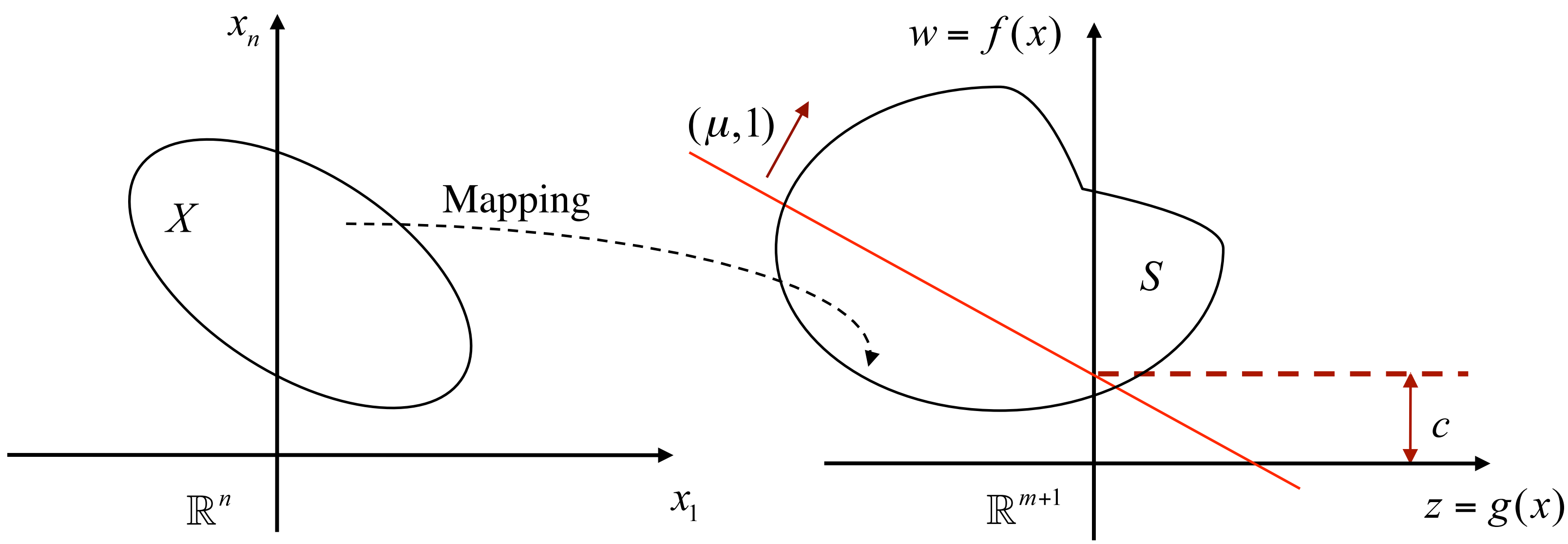


Duality Theory – A Geometric Perspective

Supporting hyperplane and dual function

Primal Problem

$$\min_{x \in X} \; f(x)$$
$$s.t. \; g(x) \le 0$$



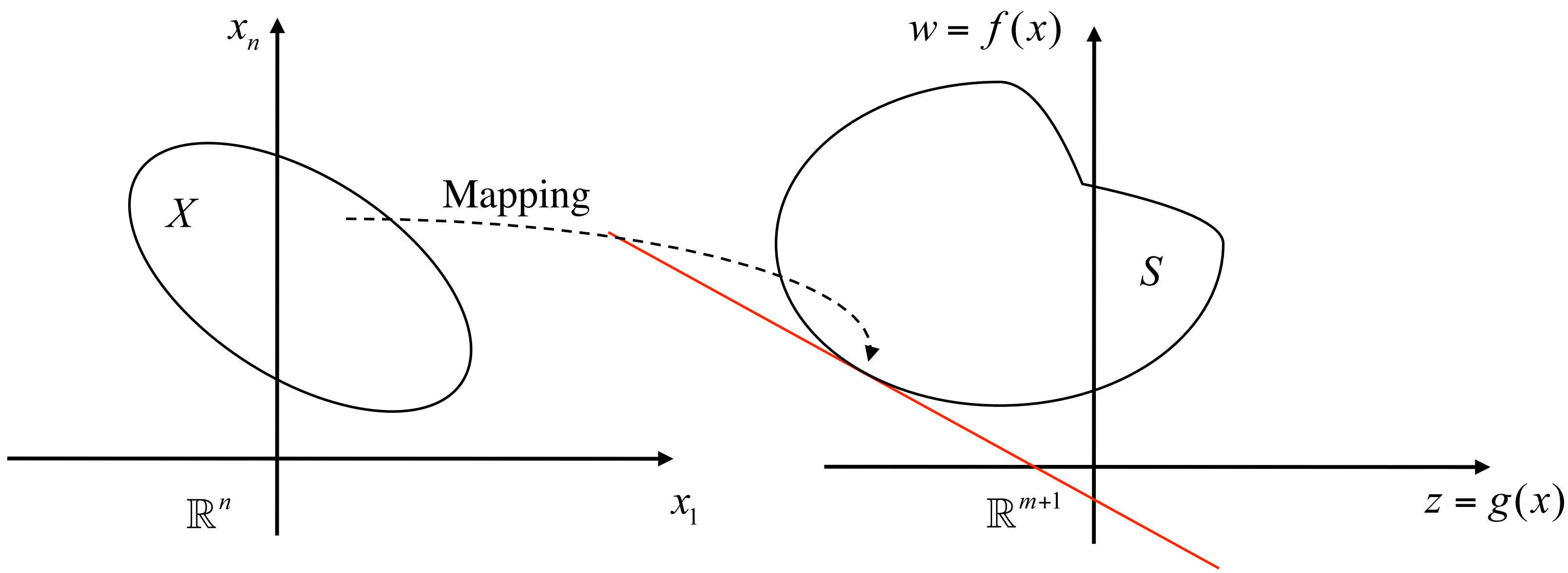
$H = \{(z,w) \in \mathbb{R}^{m+1} : \mu^T z + w = c\}$ $\qquad \mu \ge 0$

Duality Theory – A Geometric Perspective

Supporting hyperplane and dual function

Primal Problem

$$\min_{x \in X} \ f(x)$$
$$s.t. \ \ g(x) \le 0$$



$$\hat{H} = \{(z,w) \in \mathbb{R}^{m+1} : \mu^T z + w = \hat{c}\} \qquad \mu \ge 0$$

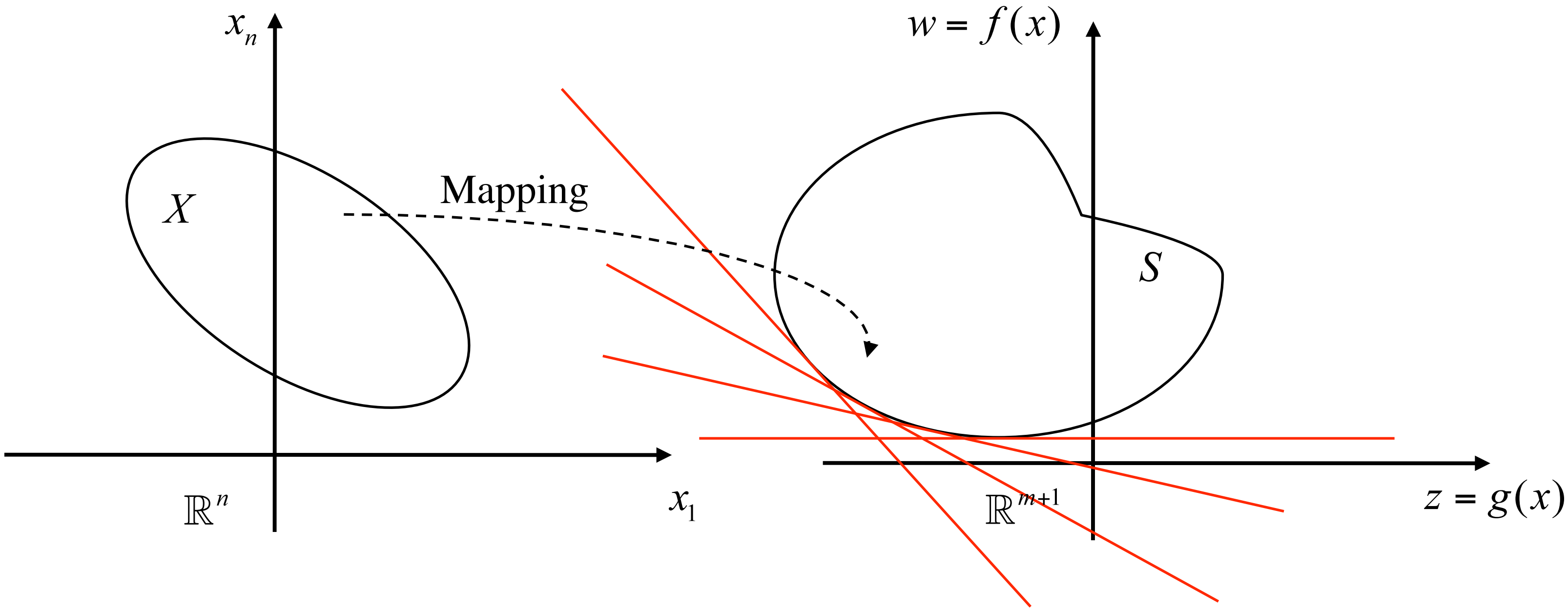
$$\hat{c} = \inf_{(z,w)\in S} \mu^T z + w = \inf_{x \in X} f(x) + \mu^T g(x) = q(\mu)$$

Dual function

Duality Theory – A Geometric Perspective

Dual problem

Primal Problem	**Dual Problem**
$\min_{x \in X} \; f(x)$	$\max_{\mu} \; q(\mu) = \inf_{x \in X} f(x) + \mu^T g(x)$
$s.t. \; g(x) \leq 0$	$s.t. \; \mu \geq 0$



$$\hat{H} = \{(z, w) \in \mathbb{R}^{m+1} : \mu^T z + w = \hat{c}\} \qquad \mu \geq 0$$

$$\hat{c} = \inf_{x \in X} f(x) + \mu^T g(x) = q(\mu)$$

Duality Theory – A Geometric Perspective

Weak duality

Primal Problem	**Dual Problem**
$\min_{x \in X} \; f(x)$	$\max_{\mu} \; q(\mu) = \inf_{x \in X} f(x) + \mu^T g(x)$
$s.t. \; g(x) \le 0$	$s.t. \; \mu \ge 0$

Weak Duality: Define $f^* = \inf_{x \in X, g(x) \le 0} f(x)$, $g^* = \sup_{\mu \ge 0} q(\mu)$, then $f^* \ge g^*$.

Proof: For all $\mu \ge 0$ and for all $x \in X$ such that $g(x) \le 0$,

$$q(\mu) = \inf_{x \in X} L(x, \mu) \le f(x) + \mu^T g(x) \le f(x),$$

so

$$q^* = \sup_{\mu \ge 0} q(\mu) \le \inf_{x \in X, g(x) \le 0} f(x) = f^*.$$

Duality Theory – A Geometric Perspective

Strong duality

Primal Problem	**Dual Problem**
$\min_{x \in X} \ f(x)$	$\max_{\mu} \ q(\mu) = \inf_{x \in X} f(x) + \mu^T g(x)$
$s.t. \ g(x) \le 0$	$s.t. \ \mu \ge 0$

Strong Duality: Let $-\infty < f^* = \inf_{x \in X, g(x) \le 0} f(x) < +\infty, \ g^* = \sup_{\mu \ge 0} q(\mu).$ If the primal problem is convex, feasible and satisfies Slater's condition, then $f^* = g^*$.

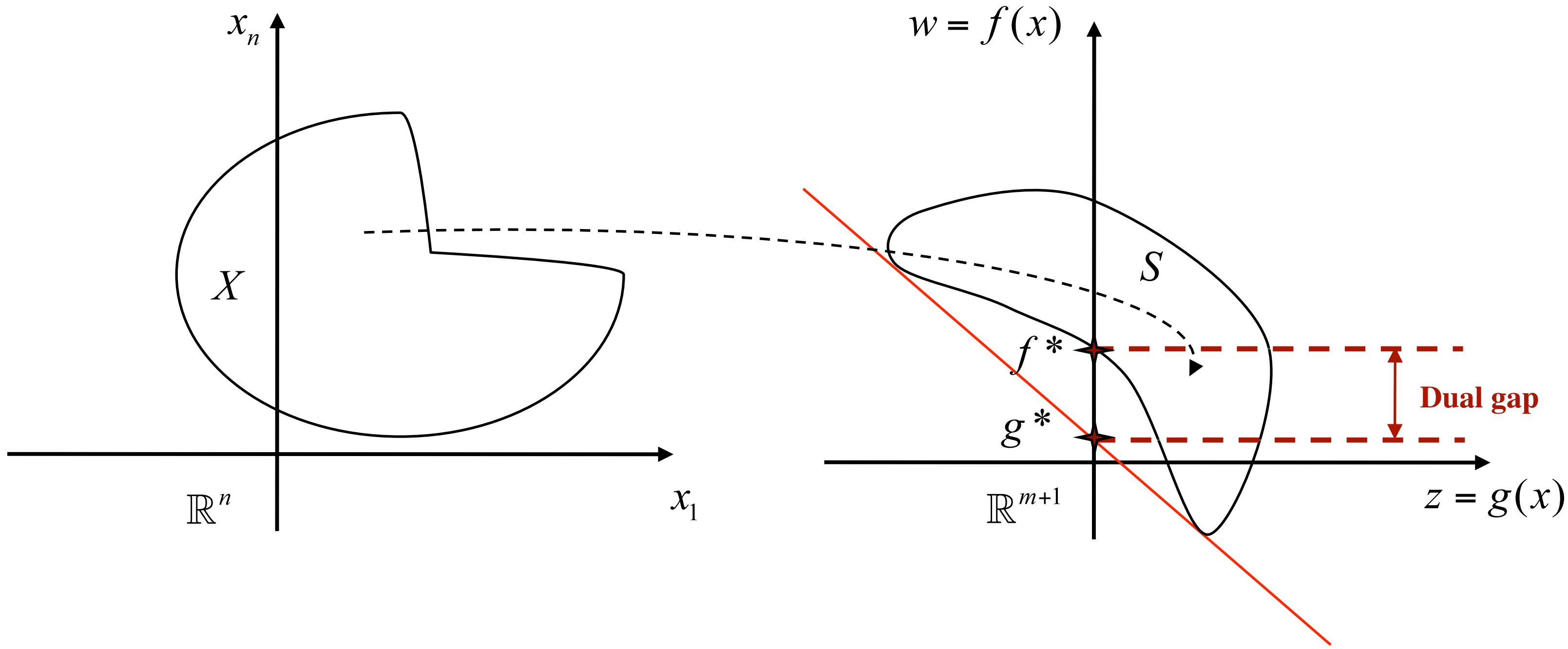
Proof: See

D. P. Bertsekas. *Nonlinear Programming*. *Athena Scientific,* 2nd edition, 1999.

Duality Theory – A Geometric Perspective

Duality gap

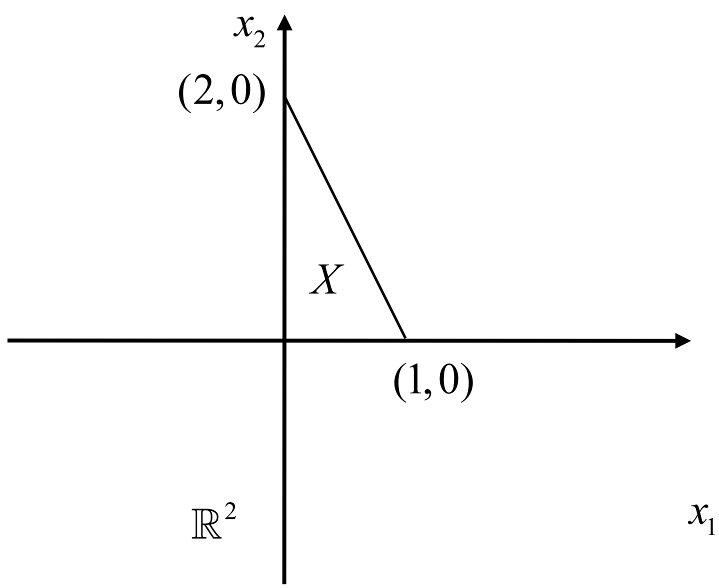
Primal Problem	**Dual Problem**
$\min_{x \in X} \; f(x)$	$\max_{\mu} \; q(\mu) = \inf_{x \in X} f(x) + \mu^T g(x)$
$s.t. \;\; g(x) \le 0$	$s.t. \;\; \mu \ge 0$

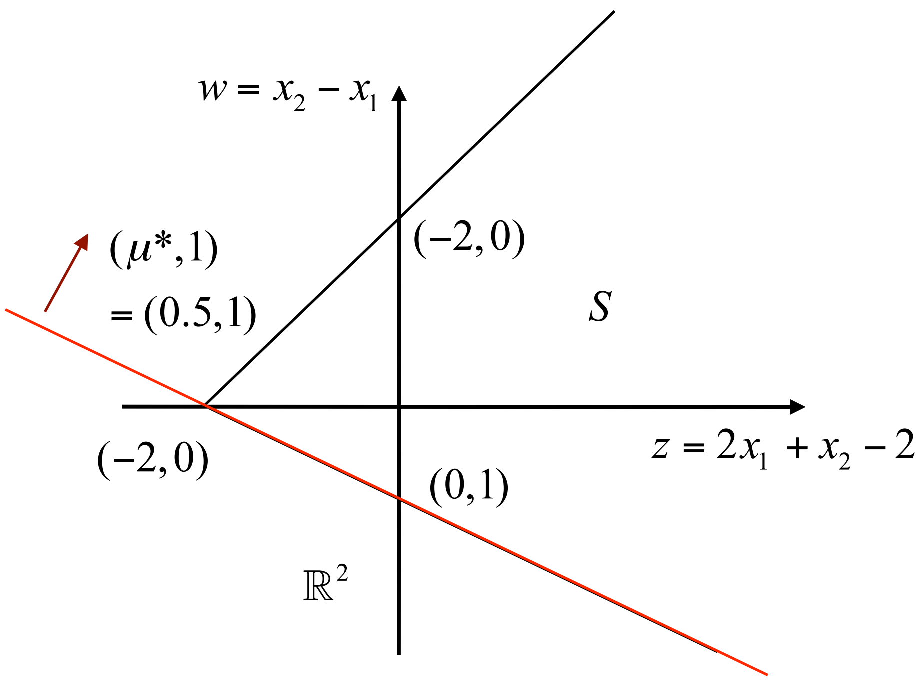


Duality Theory – A Geometric Perspective

Example 1

Primal Problem	**Dual Problem**
$\min_{x \in X} \; x_2 - x_1$	$\max_{\mu} \; \inf_{x \in X} (2\mu - 1)x_1 + (\mu + 1)x_2 - 2\mu$
$s.t. \; 2x_1 + x_2 - 2 \le 0$	$s.t. \; \mu \ge 0$
$X = \{x \in \mathbb{R}^2 : x_1 \ge 0, x_2 \ge 0\}$	





Duality Theory – A Geometric Perspective

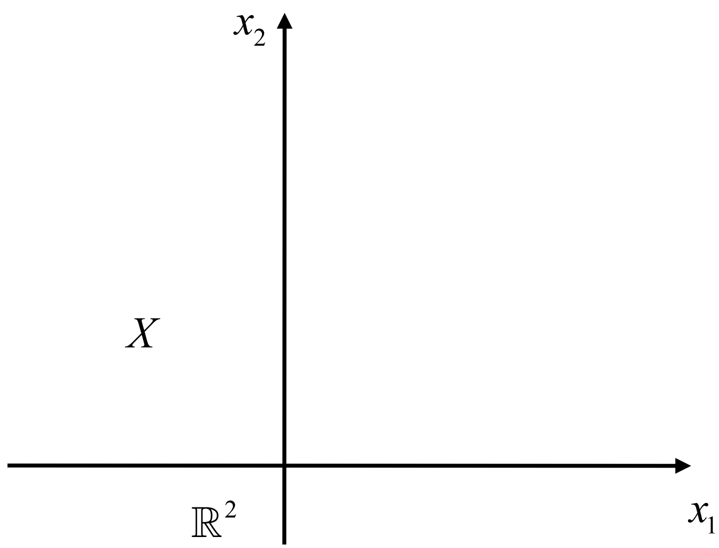
Example 2

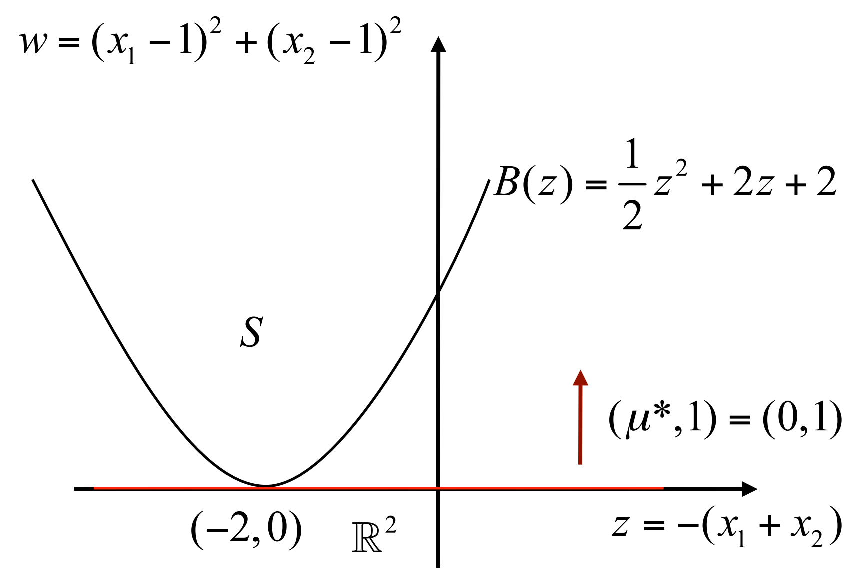
Primal Problem

$$\min_{x \in X} \ (x_1 - 1)^2 + (x_2 - 1)^2$$
$$s.t. \ -(x_1 + x_2) \le 0$$
$$X = \mathbb{R}^2$$

Dual Problem

$$\max_{\mu} \ \inf_{x \in \mathbb{R}^2} (x_1 - 1)^2 + (x_2 - 1)^2 - \mu(x_1 + x_2)$$
$$s.t. \ \mu \ge 0$$





Agenda

- Motivation
- Duality Theory – A Geometric Perspective
- Generalized Benders Decomposition
- Nonconvex Generalized Benders Decomposition
- Computational Study
- Summary

Generalized Benders Decomposition

A generalization of Benders decomposition

- Generalized Benders decomposition (GBD) is an extension of Benders decomposition.

  - A.M. Geoffrion. Generalized Benders decomposition. Journal of Optimization Theory and Applications, 10(4):237–260, 1972.

  - J. F. Benders. Partitioning procedures for solving mixed-variables programming problems. Numerische Mathematik, 4:238–252, 1962.

- Target Problem

$$
\begin{aligned}
&\min_{x,y} \; f(x,y) \\
&s.t. \; g(x,y) \le 0 \\
&\quad\;\; x \in X,\; y \in Y
\end{aligned}
$$

Assumptions: (a) $X \subset \mathbb{R}^{n_x}$ and $Y \subset \mathbb{R}^{n_y}$ are compact,

(b) For all $\hat{y} \in Y$, $f(\cdot, y) : \mathbb{R}^{n_x} \to \mathbb{R}$ and $g(\cdot, y) : \mathbb{R}^{n_x} \to \mathbb{R}^m$ are convex on X.

(c) For y fixed to any feasible $\hat{y} \in Y$, the problem satisfies Slater's condition.

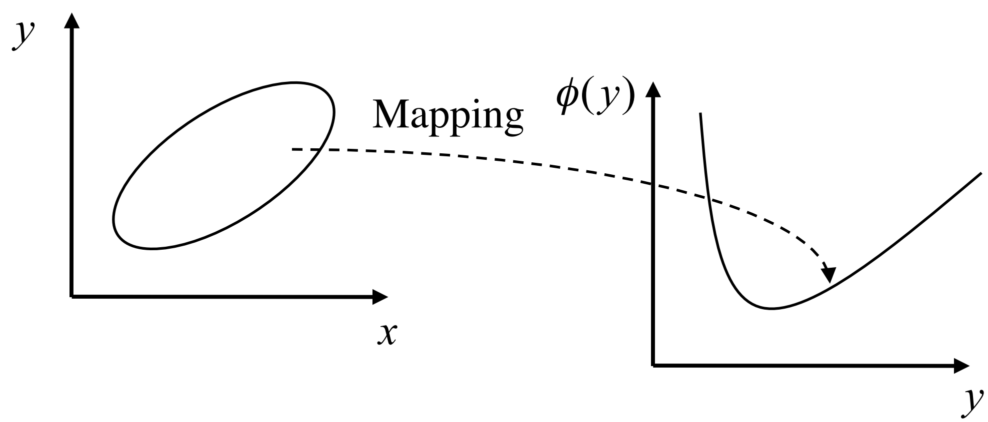
Generalized Benders Decomposition

Principle of projection

$$\min_{x,y} \; f(x,y)$$
$$s.t. \;\; g(x,y) \le 0$$
$$x \in X, \;\; y \in Y$$

$\Longleftrightarrow$

$$\min_{y} \; \phi(y)$$
$$s.t. \;\; \phi(y) = \inf_{x \in X, g(x,y) \le 0} f(x,y) \qquad \text{Optimality}$$
$$y \in Y \cap V,$$
$$V = \{y : \exists x \in X, g(x,y) \le 0\} \qquad \text{Feasibility}$$



Generalized Benders Decomposition

Optimality cuts

$$\min_{y} \ \phi(y)$$

$$s.t. \quad \phi(y) = \inf_{x \in X, g(x,y) \le 0} f(x,y) \qquad \text{Optimality}$$

$$y \in Y \cap V,$$

$$V = \{y : \exists x \in X, g(x,y) \le 0\} \qquad \text{Feasibility}$$

Generalized Benders Decomposition

Optimality

$$\min_{y} \ \phi(y)$$

$$s.t. \quad \phi(y) = \inf_{x\in X, g(x,y)\le 0} f(x,y)$$

Optimality

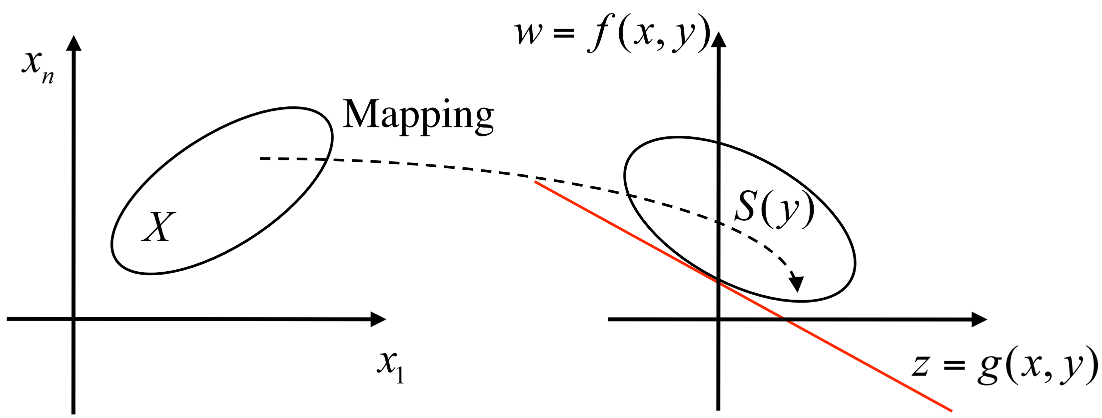
Convex, Slater's condition holds

$$\Longleftrightarrow \quad \phi(y) = \sup_{\lambda\ge 0} \inf_{x\in X} f(x,y) + \lambda^T g(x,y)$$

$$y \in Y \cap V,$$

Feasibility

$$V = \{y : \exists x \in X, g(x,y) \le 0\}$$



Generalized Benders Decomposition

Optimality, nonconvexity strikes

$$\min_y \ \phi(y)$$

$$s.t. \quad \phi(y) = \inf_{x \in X, g(x,y) \le 0} f(x,y)$$

Optimality

Convex, Slater's condition holds

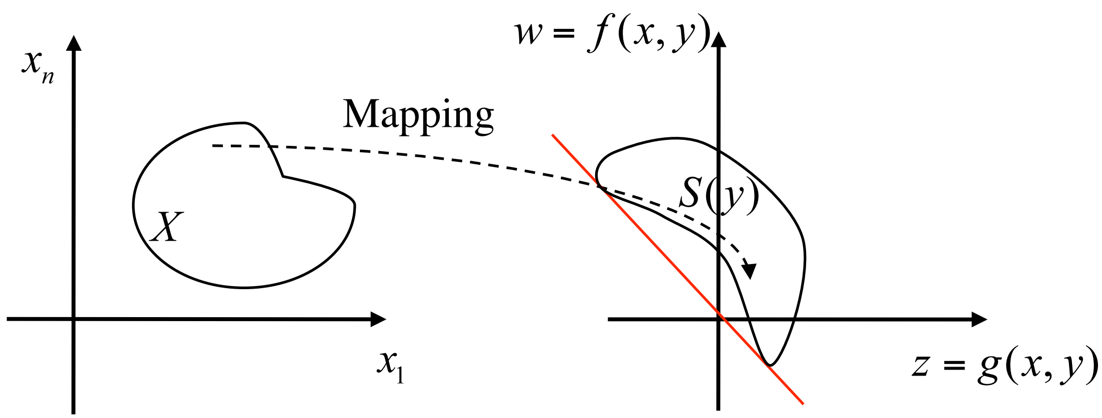
$\Longleftrightarrow$

$$\phi(y) = \sup_{\lambda \ge 0} \inf_{x \in X} f(x,y) + \lambda^T g(x,y)$$

$$y \in Y \cap V,$$

$$V = \{y : \exists x \in X, g(x,y) \le 0\}$$

Feasibility



Generalized Benders Decomposition

Feasibility

$$
\begin{aligned}
&\min_{y} \;\; \phi(y) \\
&s.t. \;\; \phi(y) = \inf_{x \in X, g(x,y) \le 0} f(x,y) && \text{Optimality} \\
&\qquad y \in Y \cap V, && \text{Feasibility} \\
&\qquad V = \{y : \exists x \in X, \, g(x,y) \le 0\}
\end{aligned}
$$

Generalized Benders Decomposition

Feasibility

$$\min_y \ \phi(y)$$

$$s.t. \quad \phi(y) = \inf_{x \in X, g(x,y) \le 0} f(x,y) \qquad \text{Optimality}$$

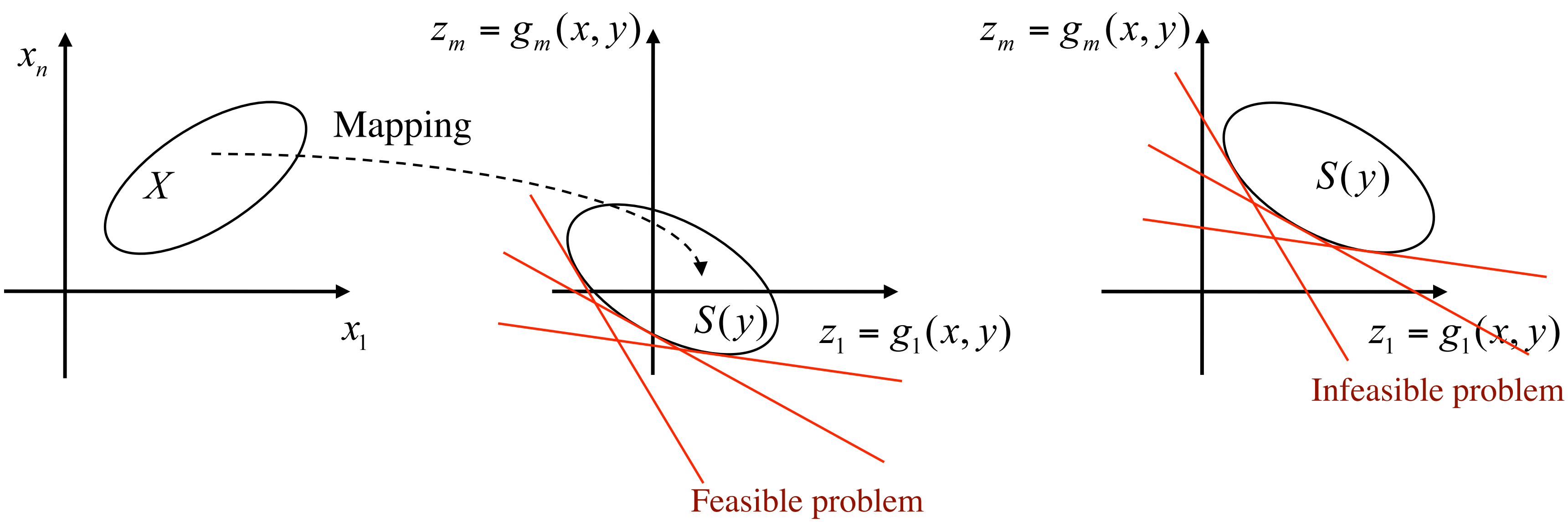
$$y \in Y \cap V,$$

$$V = \{y : \exists x \in X, g(x,y) \le 0\}$$

Feasibility **Convex** ⟺

$$\{y : 0 \ge \inf_{x \in X} \mu^T g(x,y), \quad \forall \mu \in M\}$$

$$M = \{\mu : \mu \ge 0, \sum_i \mu_i = 1\}$$



Generalized Benders Decomposition

Feasibility cuts, nonconvexity strikes

$$\min_y \ \phi(y)$$

$$s.t. \quad \phi(y) = \inf_{x \in X, g(x,y) \le 0} f(x,y)$$ Optimality

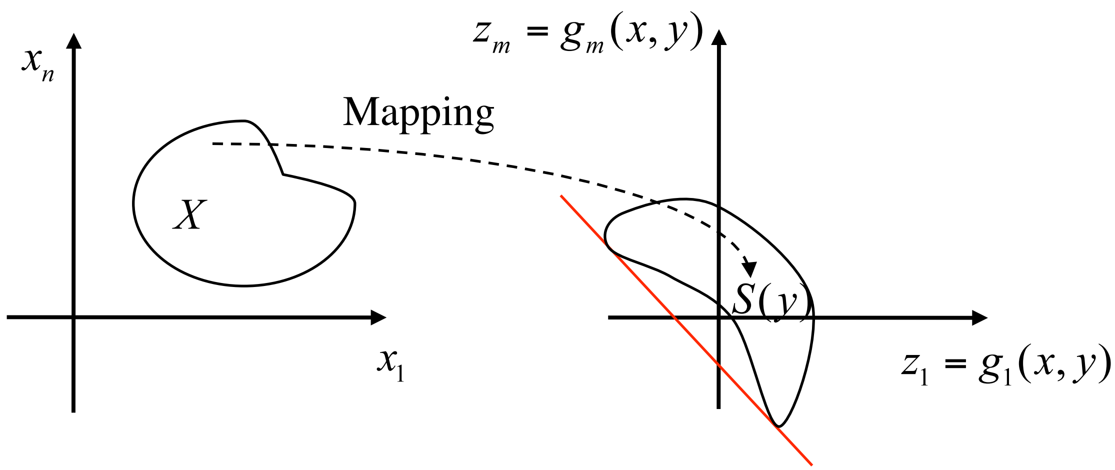
$$y \in Y \cap V,$$

$$V = \{y : \exists x \in X, g(x,y) \le 0\}$$ Feasibility

Convex ⟺

$$\{y : 0 \ge \inf_{x \in X} \mu^T g(x,y), \quad \forall \mu \in M\}$$

$$M = \{\mu : \mu \ge 0, \sum_i \mu_i = 1\}$$



Generalized Benders Decomposition

Master problem

Master Problem

$$\min_{x,y} \; f(x,y)$$
$$s.t. \; g(x,y) \le 0$$
$$x \in X, \; y \in Y$$

$$\min_{y,\eta} \; \eta$$
$$s.t. \; \eta \ge \inf_{x \in X} f(x,y) + \lambda^T g(x,y), \quad \forall \lambda \ge 0 \quad \text{Optimality cuts}$$
$$0 \ge \inf_{x \in X} \mu^T g(x,y), \quad \forall \mu \in M \quad \text{Feasibility cuts}$$

⇕ **Projection** ⇕

$$\min_{y} \; \phi(y)$$
$$s.t. \; \phi(y) = \inf_{x \in X, g(x,y) \le 0} f(x,y)$$
$$y \in Y \cap V,$$
$$V = \{y : \exists x \in X, g\left(x,y\right) \le 0\}$$

Dualization ⇔

$$\min_{y} \; \phi(y)$$
$$s.t. \; \phi(y) = \sup_{\lambda \ge 0} \inf_{x \in X} f(x,y) + \lambda^T g(x,y)$$
$$0 \ge \inf_{x \in X} \mu^T g(x,y), \quad \forall \mu \in M$$

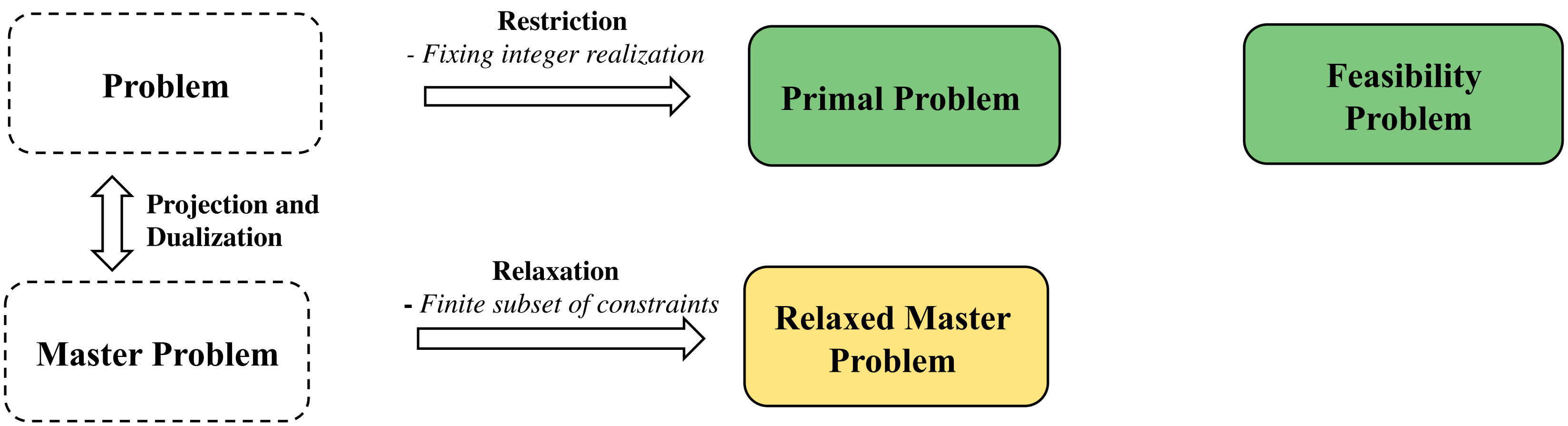
Generalized Benders Decomposition

Subproblems generated via restriction and relaxation

$$\min_{x,y} \; f(x,y)$$
$$s.t. \; g(x,y) \le 0$$
$$x \in X, \; y \in Y$$

$$obj^k_{\mathrm{Primal}} = \min_{x} \; f(x, y^{(k)})$$
$$s.t. \; g(x, y^{(k)}) \le 0$$
$$x \in X$$

$$obj^k_{Feas} = \min_{x} \; z$$
$$s.t. \; g(x, y^{(k)}) \le z,$$
$$x \in X, z \in Z \subset \{z \in \mathbb{R}^m : z \ge 0\}$$



$$\min_{y,\eta} \; \eta$$
$$s.t. \quad \eta \ge \inf_{x \in X} f(x,y) + \lambda^T g(x,y),$$
$$\forall \lambda \ge 0$$
$$0 \ge \inf_{x \in X} \mu^T g(x,y),$$
$$\forall \mu \in M$$

$$\min_{y,\eta} \; \eta$$
$$s.t. \quad \eta \ge \inf_{x \in X} f(x,y) + \left(\lambda^j\right)^T g(x,y), \quad \forall \lambda^j \in T^k$$
$$0 \ge \inf_{x \in X} \left(\mu^i\right)^T g(x,y), \quad \forall \mu^i \in S^k$$

Generalized Benders Decomposition

Relaxed master problem with separability in x and y

Relaxed Master Problem

$$\min_{y,\eta} \ \eta$$

$$s.t. \quad \eta \geq \inf_{x \in X} f(x,y) + \left(\lambda^j\right)^T g(x,y), \quad \forall \lambda^j \in T^k$$

$$0 \geq \inf_{x \in X} \left(\mu^i\right)^T g(x,y), \quad \forall \mu^i \in S^k$$

⇩

$f(x,y) \equiv f_1(x) + f_2(y), \ \ g(x,y) \equiv g_1(x) + g_2(y)$ *

This is implied in Benders decomposition!

$$\min_{y,\eta} \ \eta$$

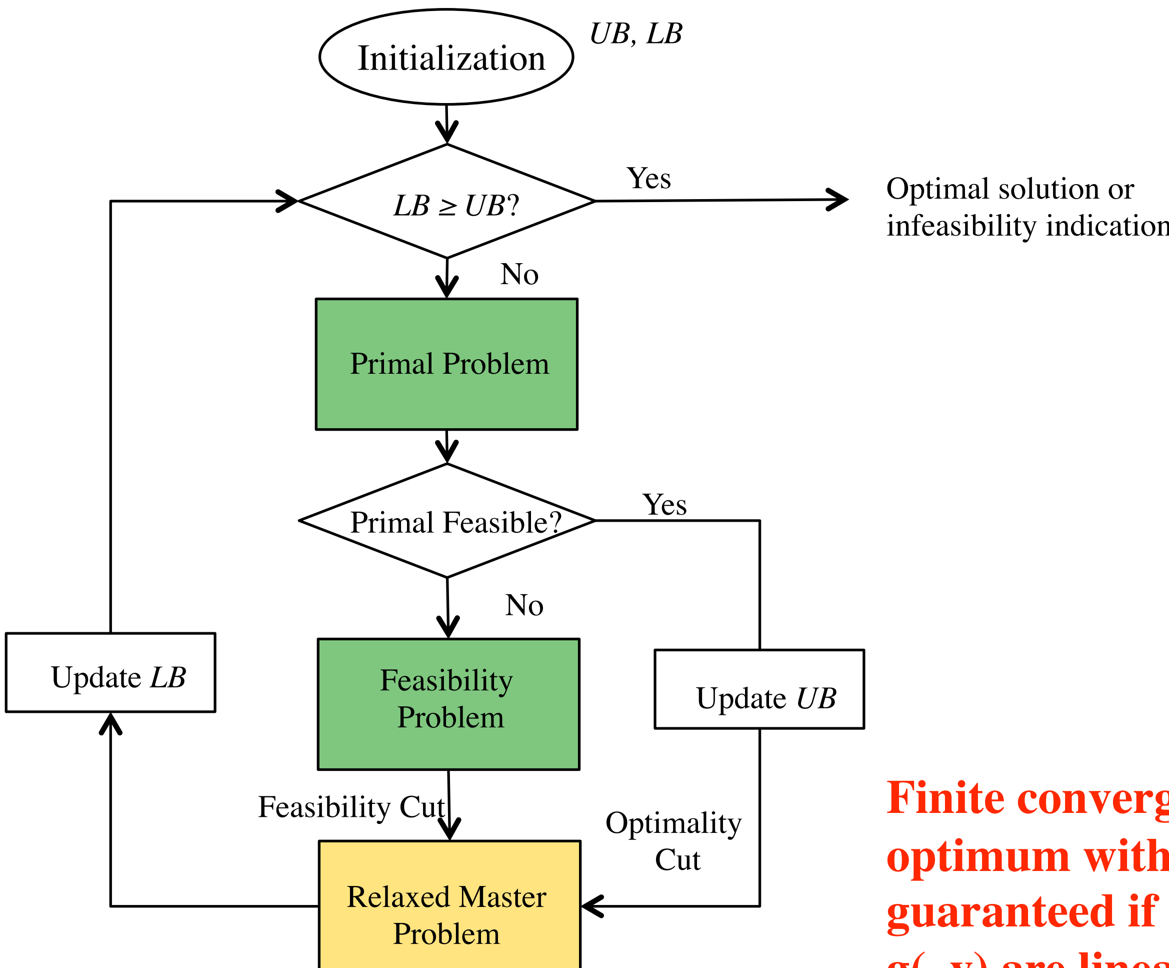
$$s.t. \quad \eta \geq obj^j_{Primal} + f_2(y) - f_2(y^{(k)}) + \left(\lambda^j\right)^T \left[g_2(y) - g_2(y^{(k)})\right], \quad \forall \lambda^j \in T^k$$

$$0 \geq obj^i_{Feas} + \left(\mu^i\right)^T \left[g_2(y) - g_2(y^{(k)})\right], \quad \forall \mu^i \in S^k$$

* A.M. Geoffrion. Generalized Benders decomposition. Journal of Optimization Theory and Applications, 10(4):237–260, 1972.

Generalized Benders Decomposition

Algorithm flowchart

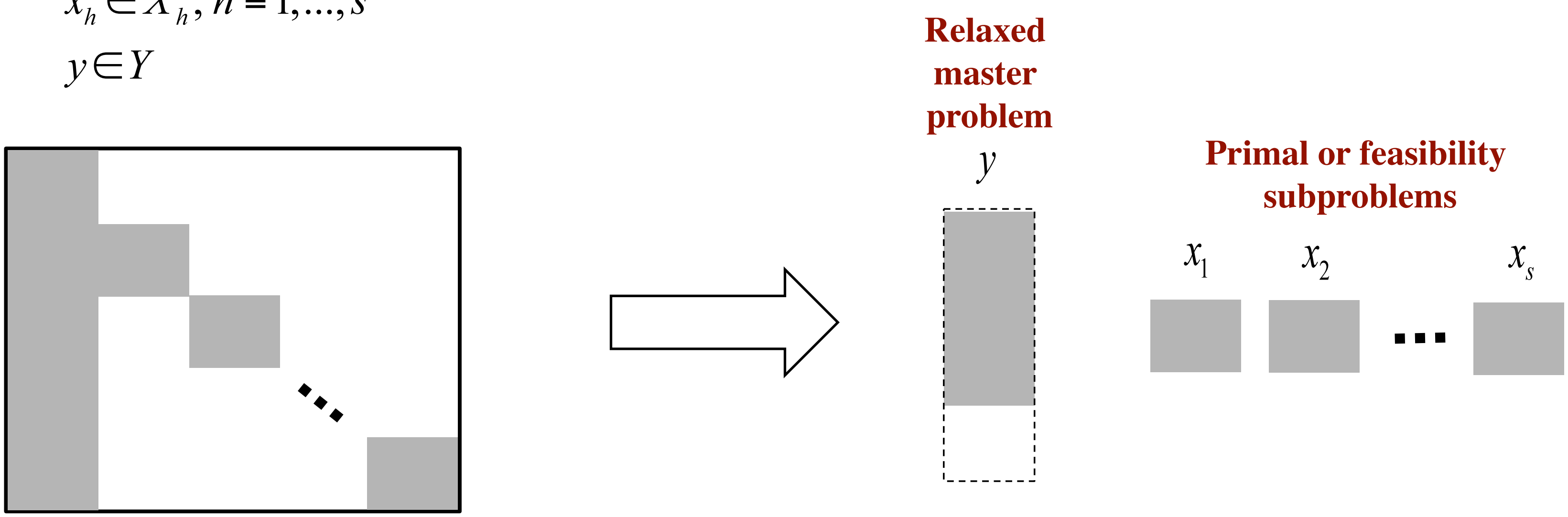


Finite convergence proof see to an optimum with a given tolerance is guaranteed if y is integer or f(.,y), g(.,y) are linear for all y in Y.

Generalized Benders Decomposition

GBD and scenario-based stochastic programs

$$
\begin{aligned}
\min_{x_1,\ldots,x_s,y} \quad & \sum_{h=1}^{s} w_h f_h(x_h, y) \\
s.t. \quad & g_h(x_h, y) \le 0, \quad h = 1,\ldots,s \\
& x_h \in X_h,\ h = 1,\ldots,s \\
& y \in Y
\end{aligned}
$$



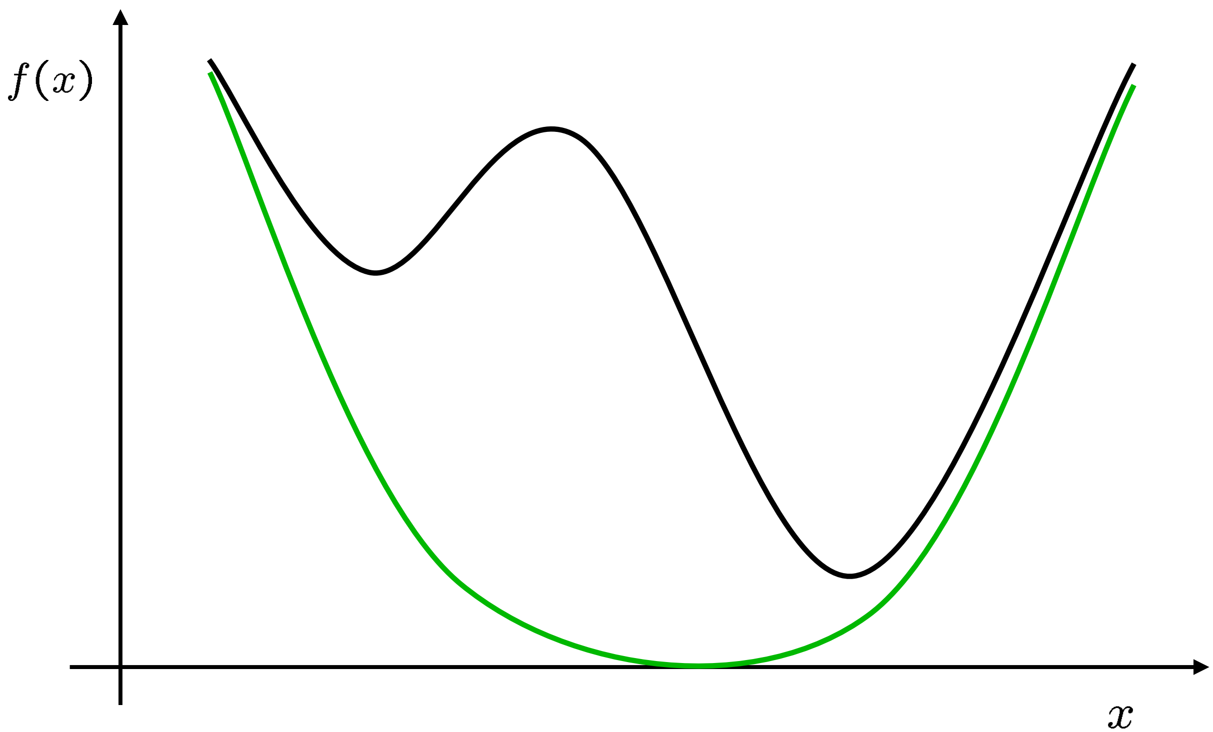
But, what if convexity assumption does not hold?

Agenda

- Motivation
- Duality Theory – A Geometric Perspective
- Generalized Benders Decomposition
- Nonconvex Generalized Benders Decomposition
- Computational Study
- Summary

Nonconvex Generalized Benders Decomposition

Convex relaxation

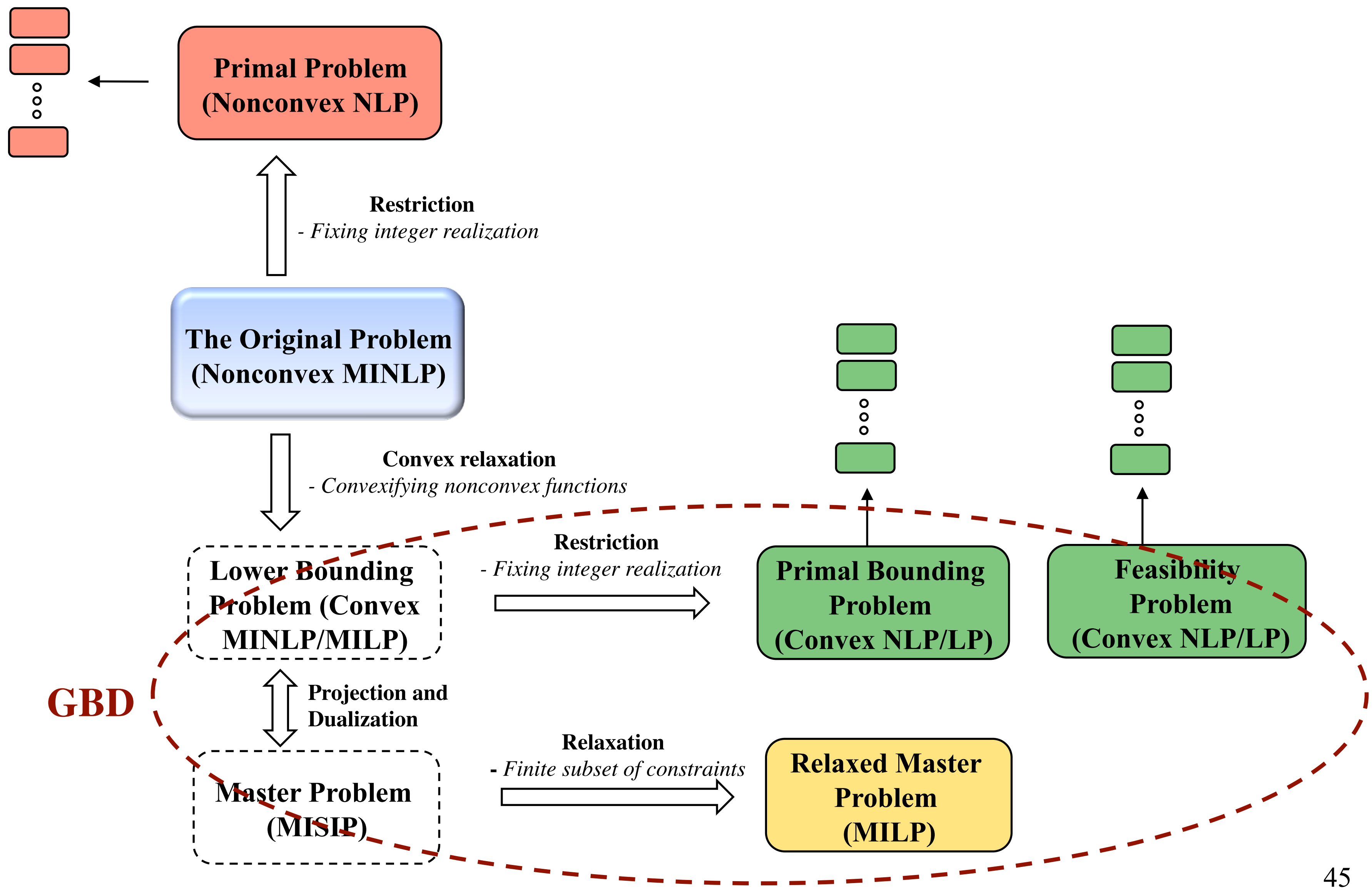


- Convex relaxation ≠ convex approximation
- Most practical problems can be relaxed via McCormick’s approach

- G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I – Convex underestimating problems. Mathematical Programming, 10:147–175, 1976.
- E. P. Gatzke, J. E. Tolsma, and P. I. Barton. Construction of convex relaxations using automated code generation technique. Optimization and Engineering, 3:305–326, 2002.

Nonconvex Generalized Benders Decomposition

Overview



Nonconvex Generalized Benders Decomposition

Lower bounding problem via convexification

The Original Problem (Nonconvex MINLP)

$$\min_{x_1,\dots,x_s,y} \sum_{h=1}^{s} w_h \left(f_h(x_h) + c_h^T y\right)$$
$$s.t.\ \ g_h(x_h) + B_h y \le 0, \quad h = 1,\dots,s$$
$$x_h \in X_h, \quad h = 1,\dots,s$$
$$y \in Y$$

Lower Bounding Problem (Convex MINLP/MILP)

$$\min_{x_1,\dots,x_s,y} \sum_{h=1}^{s} w_h \left(u_{f,h}(x_h, q_h) + c_h^T y\right)$$
$$s.t.\ \ u_{g,h}(x_h, q_h) + B_h y \le 0, \quad h = 1,\dots,s$$
$$x_h \in X_h,\ q_h \in Q_h, \quad h = 1,\dots,s$$
$$y \in Y$$

Nonconvex Generalized Benders Decomposition

Primal bounding problem and feasibility problem

Primal Bounding Problem (LP)

$$
\begin{aligned}
obj_{PBP_h}^{(k)} = \min_{x_h, q_h} \quad & w_h \left(u_{f,h}(x_h, q_h) + c_h^T y^{(k)} \right) \\
s.t. \quad & u_{g,h}(x_h, q_h) + B_h y^{(k)} \le 0, \\
& (x_h, q_h) \in X_h \times Q_h
\end{aligned}
$$

Feasibility Problem (LP)

$$
\begin{aligned}
obj_{FP_h}^{(k)} = \min_{x_h, q_h, z_h} \quad & w_h \| z_h \| \\
s.t. \quad & u_{g,h}(x_h, q_h) + B_h y^{(k)} \le z_h, \\
& (x_h, q_h) \in X_h \times Q_h, \\
& z_h \in Z \subset \{ z \in \mathbb{R}^m : z \ge 0 \}
\end{aligned}
$$

Nonconvex Generalized Benders Decomposition

Master problem

Master Problem (MISIP)

$$\min_{\eta, y} \quad \eta$$

$$s.t. \ \eta \geq \sum_{h=1}^{s} \left(w_h c_h^T + \lambda_h^T B_h \right) y + \sum_{h=1}^{s} \inf_{(x_h, q_h) \in X \times Q} \left[w_h u_{f,h}(x_h, q_h) + \lambda_h^T u_{g,h}(x_h, q_h) \right],$$

$$\forall \lambda_1, \ldots, \lambda_s \geq 0,$$

$$0 \geq \sum_{h=1}^{s} \mu_h^T B_h y + \sum_{h=1}^{s} \inf_{(x_h, q_h) \in X \times Q} \mu_h^T u_{g,h}(x_h, q_h),$$

$$\forall \mu_1, \ldots, \mu_s \in M,$$

$$y \in Y, \eta \in \mathbb{R}$$

Nonconvex Generalized Benders Decomposition

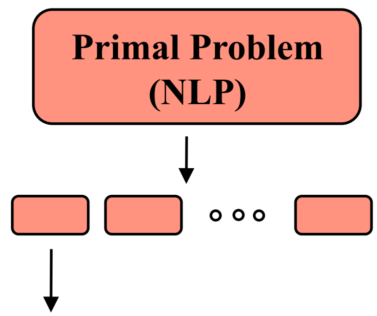
Relaxed master problem

Relaxed Master Problem (MILP)

$$
\begin{aligned}
\min_{\eta, y} \quad & \eta \\
s.t. \;\; & \eta \geq obj_{PBP}^{(j)} + \left(\sum_{h=1}^{s} \left(w_h c_h^T + \left(\lambda_h^{(j)}\right)^T B_h \right) \right) \left(y - y^{(j)} \right), \quad \forall j \in T^k, \\
& 0 \geq obj_{FP}^{(i)} + \left(\sum_{h=1}^{s} \left(\mu_h^{(i)}\right)^T B_h \right) \left(y - y^{(i)} \right), \quad \forall i \in S^k, \\
& \sum_{r \in \Xi^{(1)}} y_r - \sum_{r \in \Xi^{(0)}} y_r \leq \left| \{ r : y_r^{(t)} = 1 \} \right| - 1, \quad \forall t \in T^k \cup S^k, \\
& y \in Y, \eta \in \mathbb{R}
\end{aligned}
$$

Nonconvex Generalized Benders Decomposition

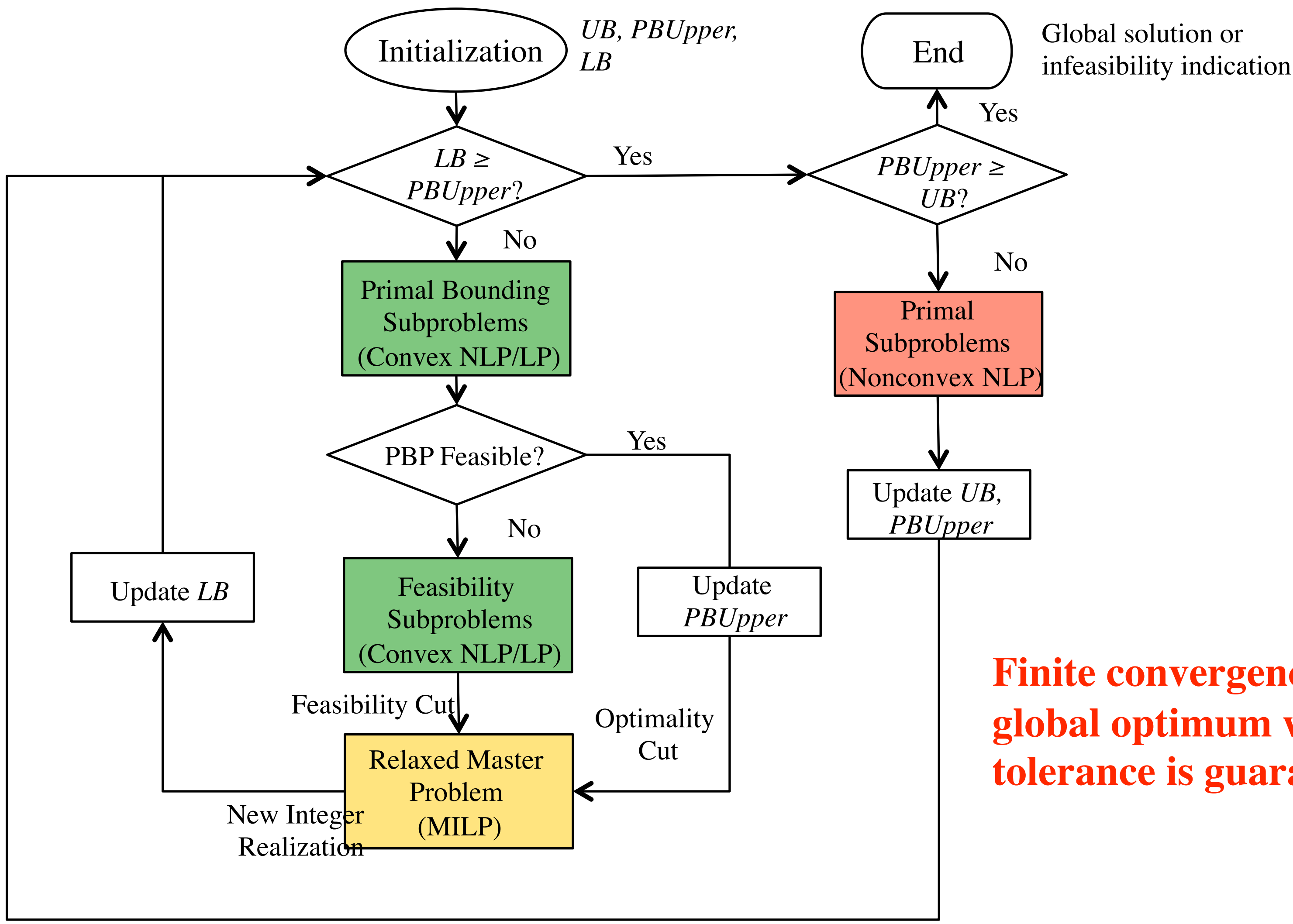
Primal problem



$$
\begin{aligned}
obj_{PP_h}^{(l)} = \min_{x_h} \;& w_h \left(f_h(x_h) + c_h^T y^{(l)} \right) \\
s.t. \;& g_h(x_h) + B_h y^{(l)} \le 0, \\
& f_h(x_h) + c_h^T y^{(l)} \le UBD_h^{(l)}, \\
& x_h \in X_h
\end{aligned}
$$

Nonconvex Generalized Benders Decomposition

Algorithm flowchart



Finite convergence to a global optimum with a given tolerance is guaranteed!

Agenda

- Motivation
- Duality Theory – A Geometric Perspective
- Generalized Benders Decomposition
- Nonconvex Generalized Benders Decomposition
- Computational Study
- Summary

Computational Study
- Implementation Issues

Platform

- CPU 2.83 GHz, Memory 1 GB, Linux, GAMS 22.8.1

Solvers

- LP and MILP solver : CPLEX
- Global NLP solver: BARON
- Local NLP solver: SNOPT

Methods for Comparison

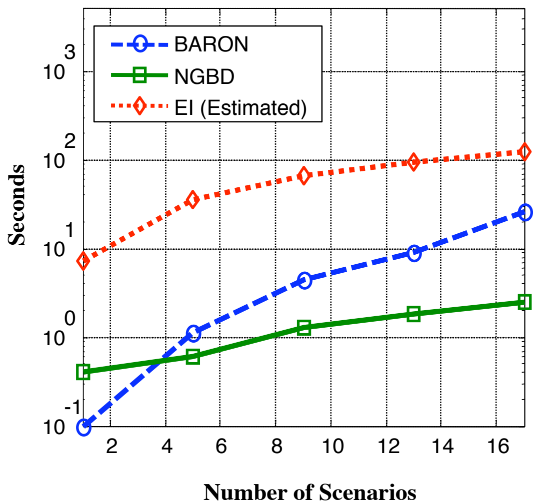
1. BARON – The state-of-the-art global optimization solver
2. DA – The proposed decomposition method
3. EI – Naïve integer enumeration

Relative Tolerance for Global Optimization

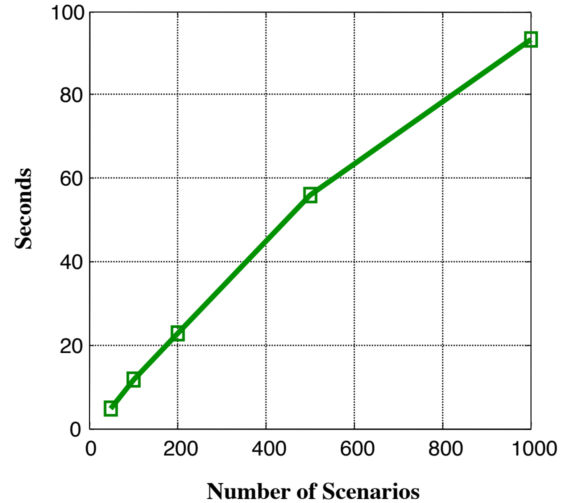
- 10^{-2}

Computational Study
- The Stochastic Haverly Pooling Problem

The stochastic problem contains 16 binary variables and 21s continuous variables (s represents total number of scenarios).

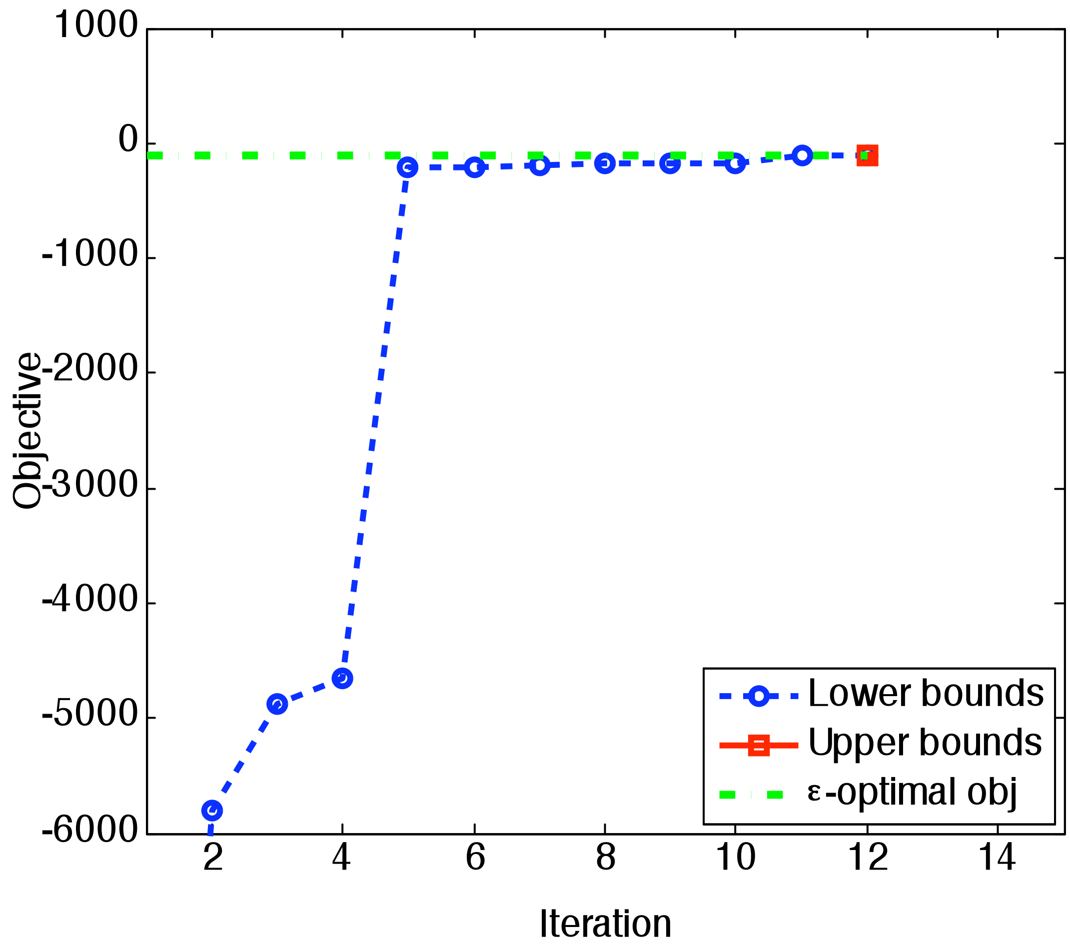


(a) Solver times with different methods



(b) Solver times with NGBD for more scenarios

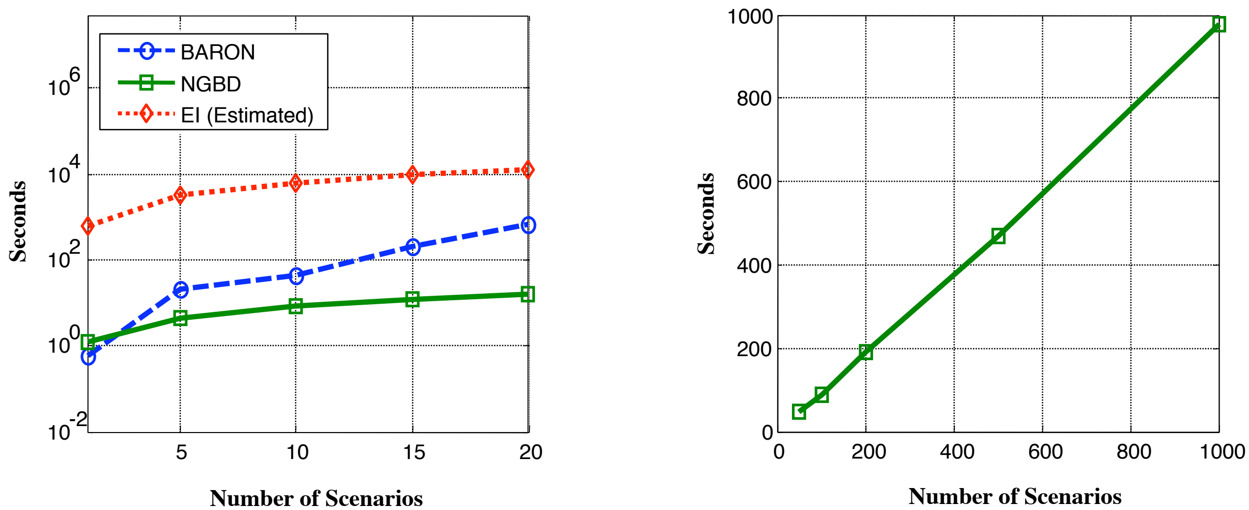
Computational Study
- The Stochastic Haverly Pooling Problem



Convergence of the upper and lower bounds over the iterations

Computational Study
- SGPS design problem A

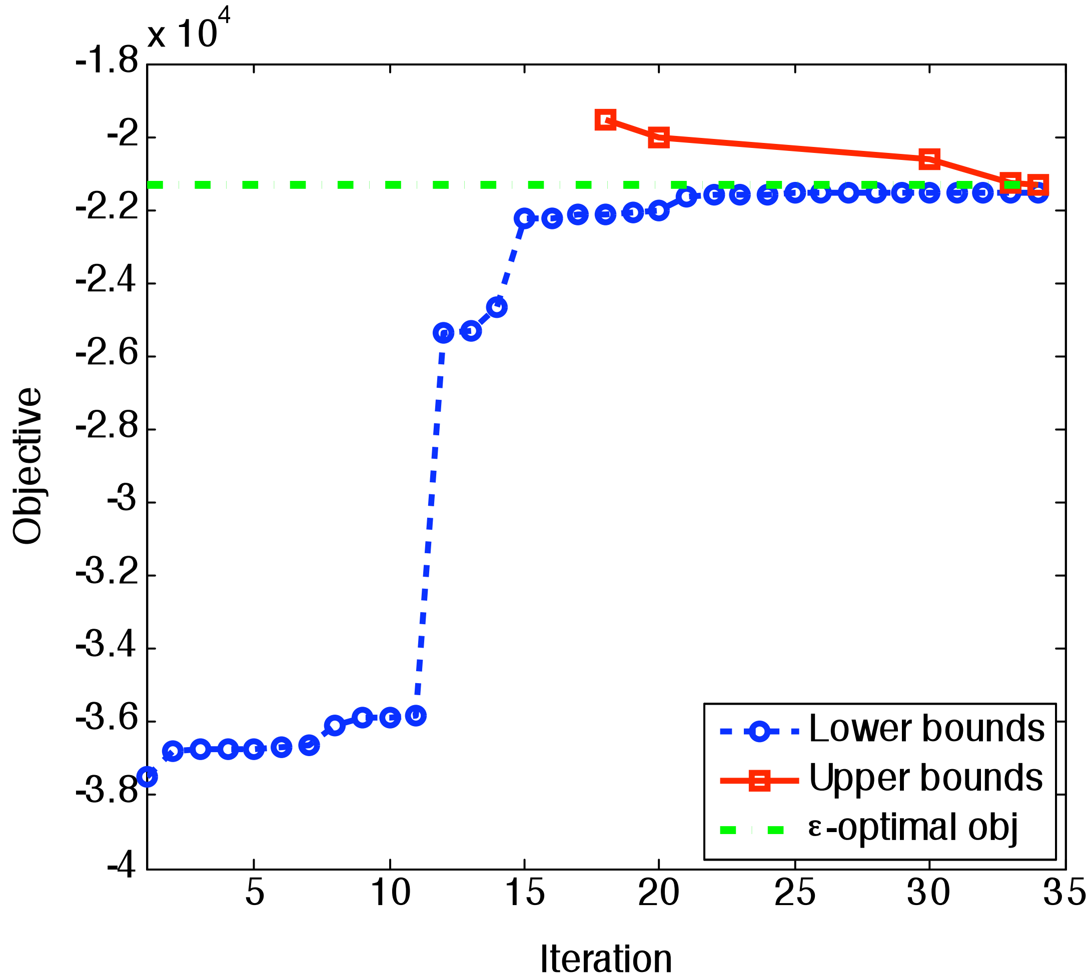
The stochastic problem contains 38 binary variables and 93s continuous variables (s represents total number of scenarios).



(a) Solver times with different methods

(b) Solver times with NGBD for more scenarios

Computational Study
- SGPS design problem A



Convergence of the upper and lower bounds over the iterations

Computational Study
- Summary of studied nonvonex problems (all solved to global optimality with given tolerances)

	Continuous variable /Integer variable	Time via NGBD (Second)	Nonconvexity	Implementation [1]
Haverly	21,000/16	93.4	Bilinear	(A)
Gas Network	68,000/19	15,610.7	Bilinear	(A)
SGPS A	93,000/38	978.2	Bilinear	(A)
SGPS B	93,000/38	977.1	Bilinear	(A)
SGPS C	146,410/20	4,234.8	Bilinear, quadratic, power	(B)
Software	10,648/8	260.7	Logarithmic	(B)
Pump	50,578/18	2,794.8	Bilinear, quadratic, cubic	(B)
Polygeneration	14,688/70	15,825.0 [2]	Bilinear	(C)

Note: [1] Problems were run with different CPUs, GAMS systems and relative termination tolerances δ:
(A) CPU 2.83 GHz , GAMS 28.1, $\delta=10^{-2}$; (B) CPU 2.83 GHz, GAMS 23.4, $\delta=10^{-3}$; (C) CPU 2.66 GHz, GAMS 23.5, $\delta=10^{-2}$.
[2] Enhanced NGBD with tighter lower bounding problems employed.

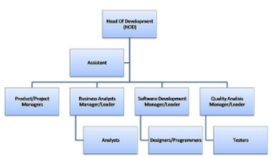
Agenda

- Motivation
- Duality Theory – A Geometric Perspective
- Generalized Benders Decomposition
- Nonconvex Generalized Benders Decomposition
- Computational Study
- Summary

Summary

- Energy infrastructure design problems may be modeled as **stochastic separable mixed-integer-nonlinear programs**;
- **Generalized Benders decomposition** provides a promising nonlinear decomposition framework for scenario-based stochastic programs;
- **Nonconvex generalized Benders decomposition** tackles nonconvexity via relaxation to convex surrogate problems;
- **Nonconvex** engineering problems with practical sizes have been solved with NGBD to **global optimality** within reasonable time.

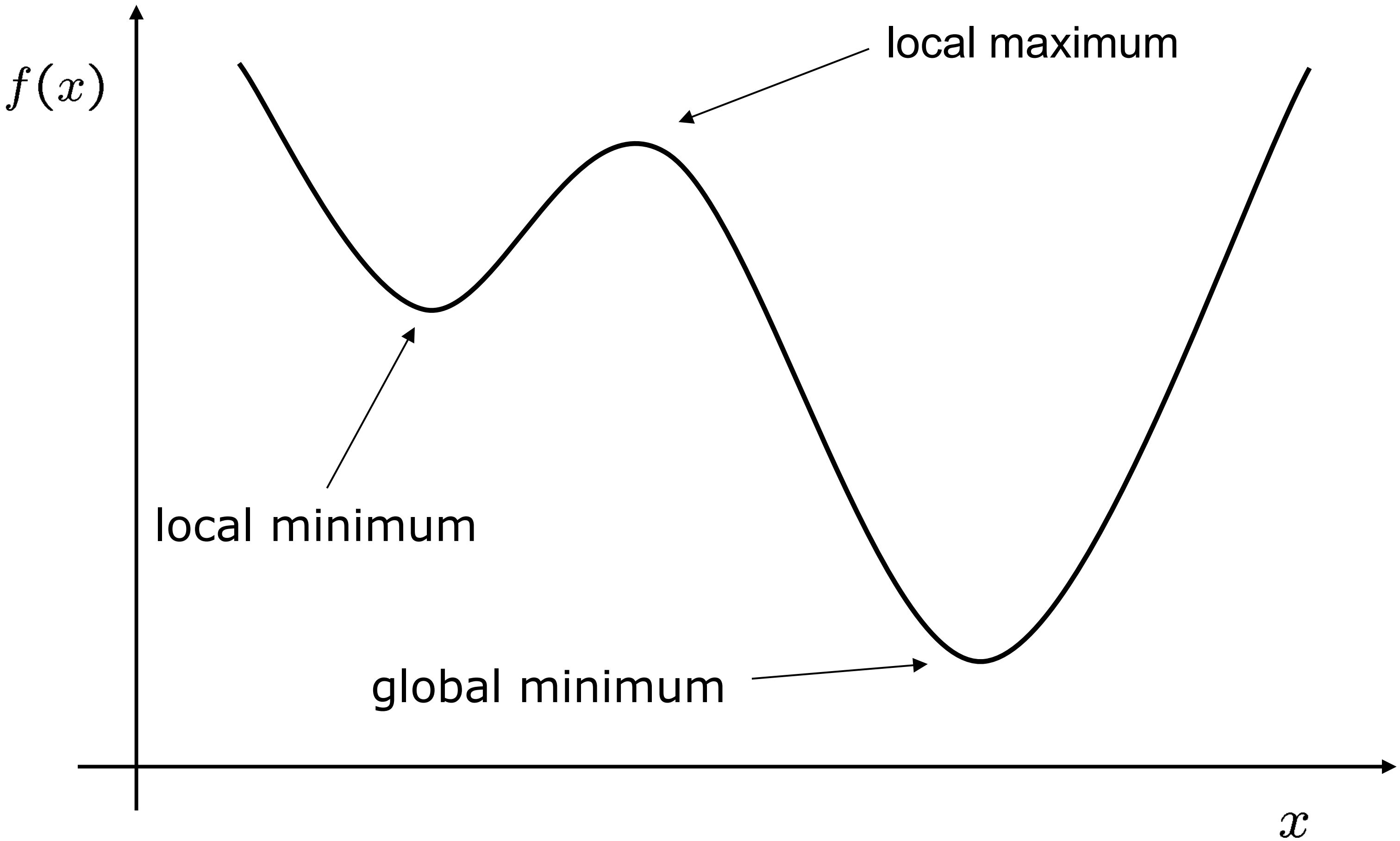
The end, and not the end.



$$\min_{x_1,\ldots,x_s,y} \sum_{h=1}^{s} w_h \left(f_h(x_h) + c_h^T y \right)$$
$$s.t.\ \ g_h(x_h) + B_h y \le 0, \quad h = 1,\ldots,s$$
$$x_h \in X_h, \quad h = 1,\ldots,s$$
$$y \in Y$$

Global Optimization via Branch-and-Bound

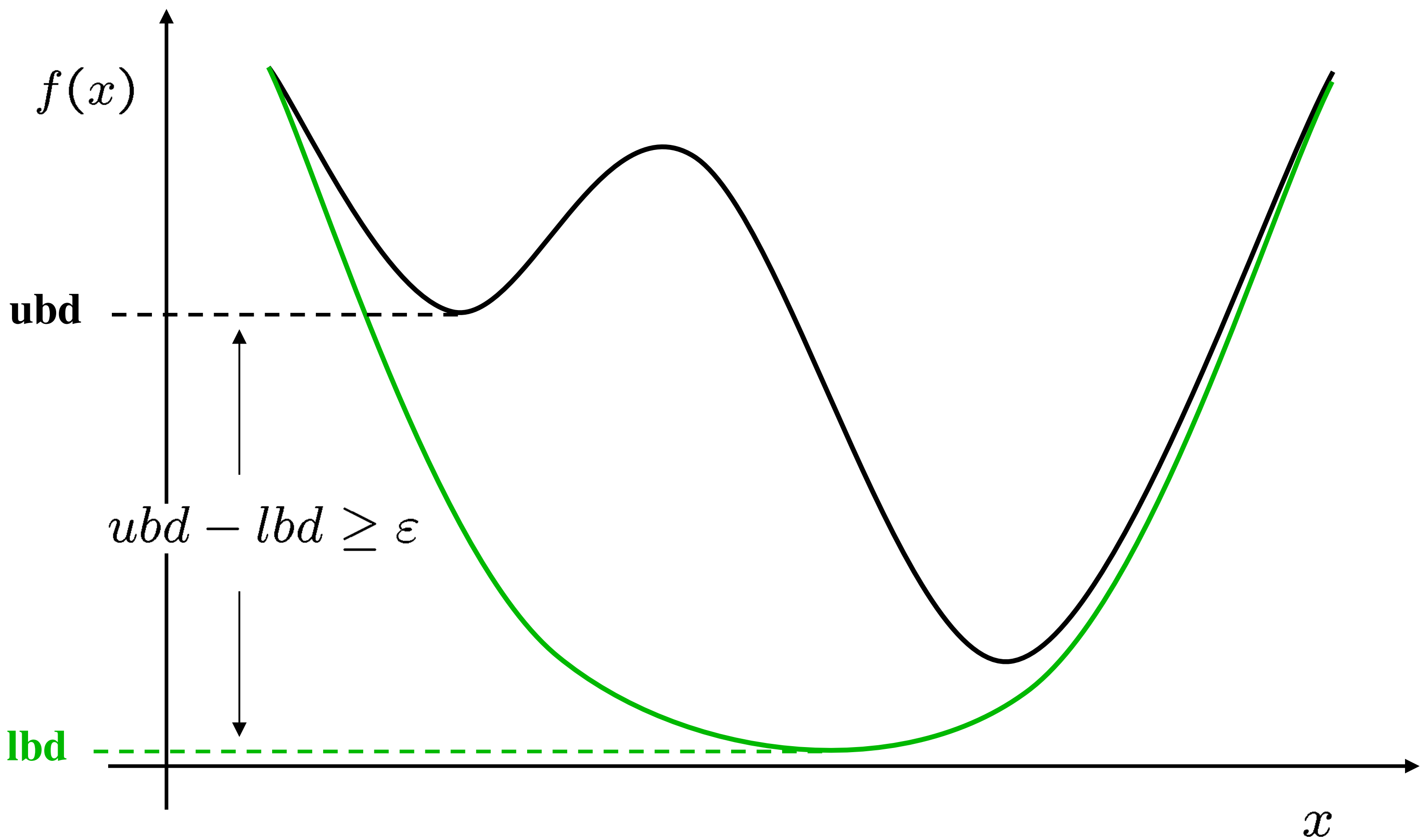
- Nonconvex optimization



Standard optimization techniques cannot distinguish between suboptimal local minima

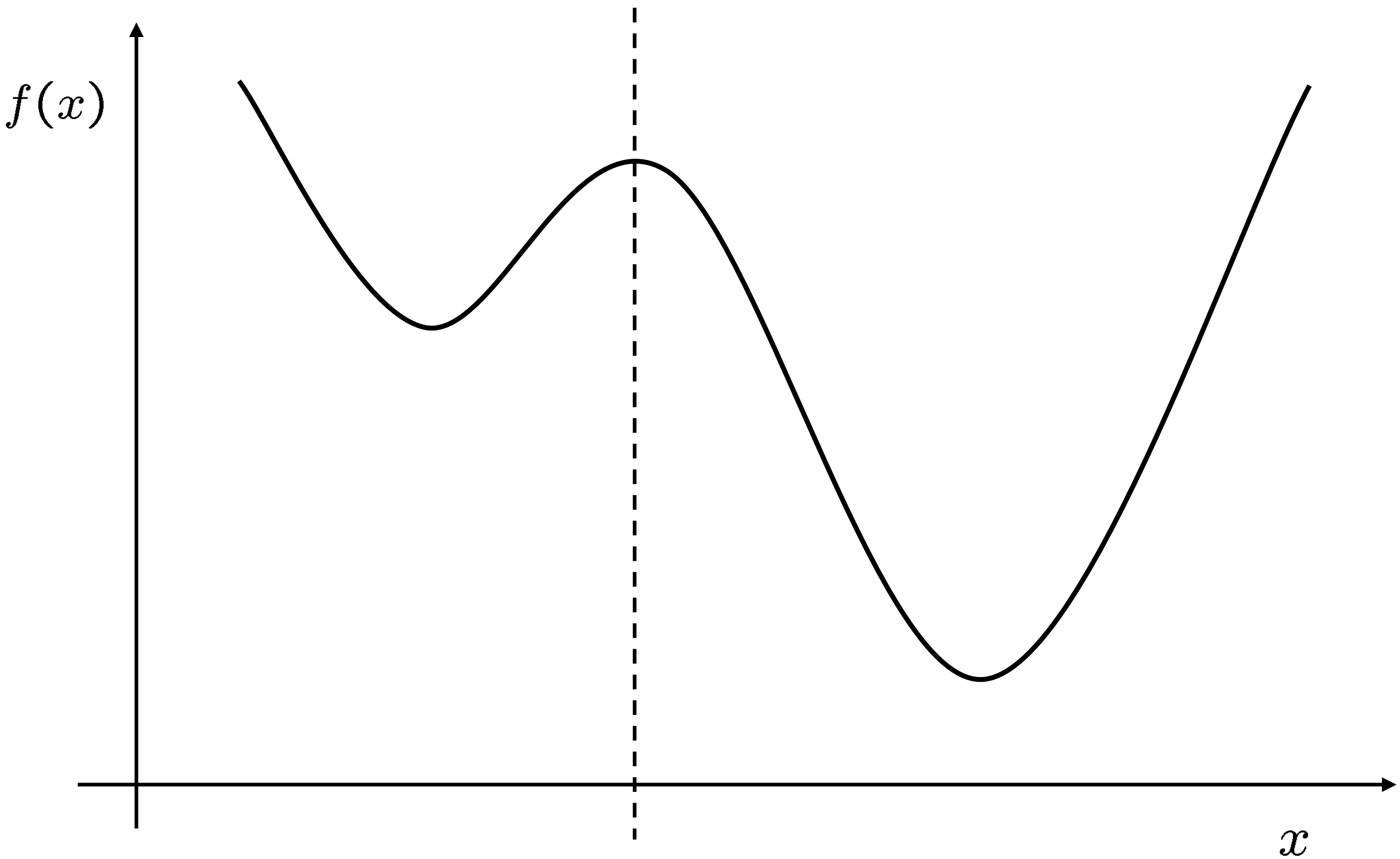
Global Optimization via Branch-and-Bound

- Convex Relaxation



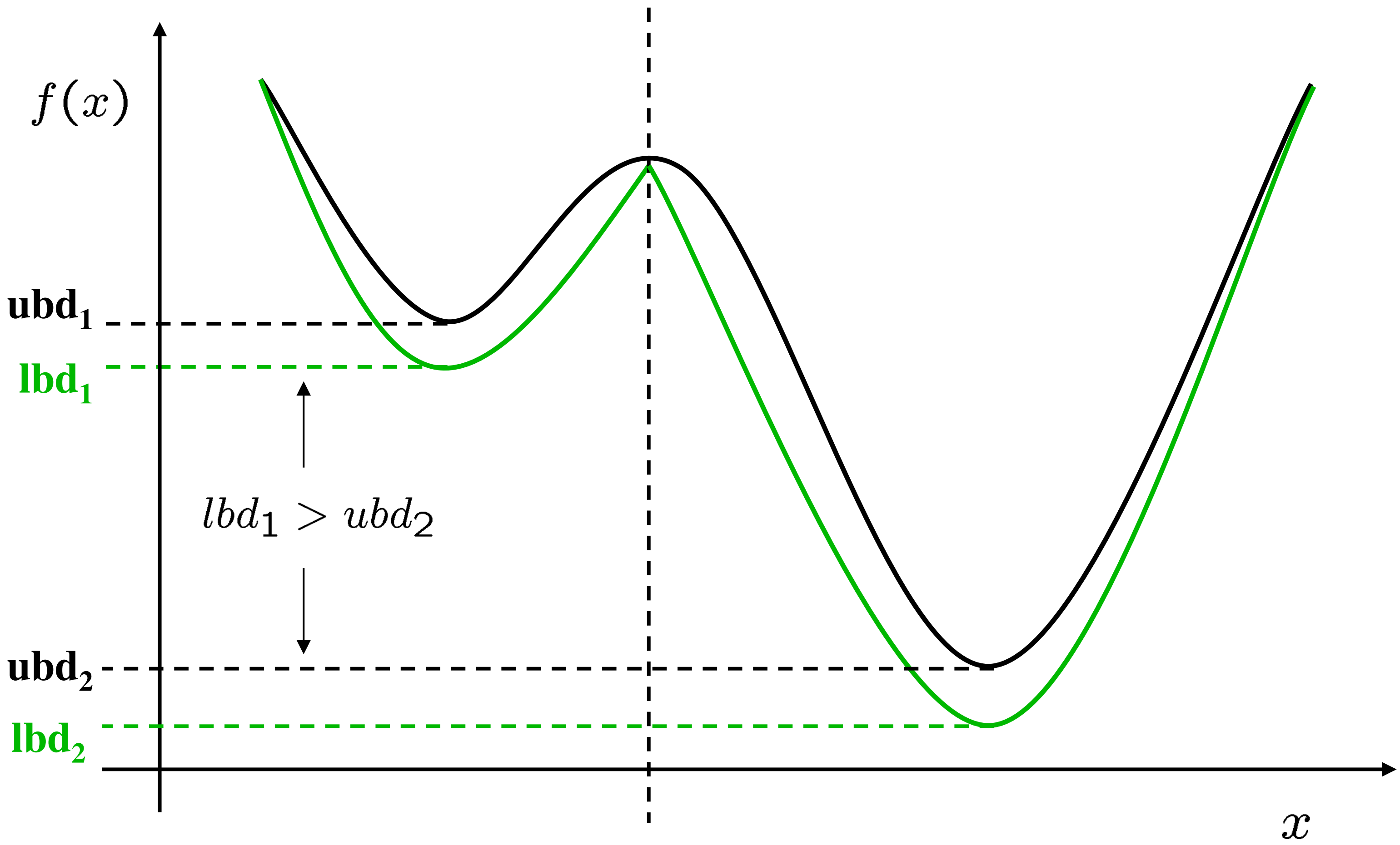
Global Optimization via Branch-and-Bound

- Branch



Global Optimization via Branch-and-Bound

- Branch, and bound



Global Optimization via Branch-and-Bound

- Branch, bound and fathom